IEOR 4106: HMWK 3

1. Each of the following transition matrices is for a Markov chain. For each, find the communication classes for breaking down the state space, \( S = C_1 \cup C_2 \cup \cdots \).

(a)
\[
P = \begin{pmatrix}
  \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
  \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\]

(b)
\[
P = \begin{pmatrix}
  0 & 0 & \frac{1}{2} & \frac{1}{2} \\
  0 & 0 & \frac{1}{2} & \frac{1}{2} \\
  0 & 0 & \frac{1}{2} & \frac{1}{2} \\
  0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

(c)
\[
P = \begin{pmatrix}
  \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
  0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
  \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{3}{5} \\
  0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
  \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2}
\end{pmatrix}.
\]

(d)
\[
P = \begin{pmatrix}
  \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
  \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
  \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
  0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
  \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3}
\end{pmatrix}.
\]

2. Consider a Markov chain with state space \( S = \{0, 1, 2, 3\} \) with transition matrix
\[
P = \begin{pmatrix}
  \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\
  \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
  \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\
  0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\]

(a) Show that all states communicate, hence the chain has only one communication class, \( C = S = \{0, 1, 2, 3\} \) hence is irreducible.

(b) Solve \( \pi = \pi P \) for the limiting distribution \( \pi = (\pi_0, \pi_1, \pi_2, \pi_3) \), where \( \pi_j > 0 \), \( j \in S \), and \( \sum_{j \in S} \pi_j = 1 \).

(c) Given that the chain is now in state 2, what is the expected amount of time until the chain returns to state 2?
(d) If $X_n$ denotes the amount of bonus money you earn during the $n^{th}$ year of your job (in units of 10,000) then, on average (over all time) what is your average bonus?

3. George has 3 umbrellas distributed between home and office as follows: When departing home at the beginning of a day, if it is raining, then he takes an umbrella (if there is one) with him to the home. Similarly, when departing the office the end of a day, if it is raining, then he takes an umbrella (if there is one) with him to the office. Assume that independent of the past there is a fixed probability $0 \leq p \leq 1$ that it rains any time he departs a location (home or office).

(a) Argue that $X_n$ is the number of umbrellas at the current location JUST BEFORE he departs for the $n^{th}$ time forms a MC, with state space $S = \{0,1,2,3\}$ and find the transition matrix.

(b) What is the long-run proportion of times that George gets wet?

(c) What value of $p$ maximizes the long-run proportion of times that George gets wet? Using this value, compute the average number of umbrellas just before he departs.

(d) More generally, when George has $r \geq 3$ umbrellas, find the transition matrix; the state space is then $S = \{0,1,2,\ldots, r\}$. Verify (simply plug in and check that it works), that the solution is as given on Page 272 of Text, Problem 46 (ii).

4. Martingale MC: Consider the MC with state space $S = \{0,1,2,3,4\}$ and transition matrix

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
0 & 3/5 & 0 & 1/5 & 1/5 \\
1/10 & 1/10 & 1/10 & 1/10 & 6/10 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.$$ 

(a) Show that $E(X_{n+1}|X_n = i) = i$, $i \in S$. (e.g., each row $i$ of $P$ has mean $i$.) Thus, being so for each $i$, we conclude that $E(X_{n+1}|X_n) = X_n$: Given the present state $X_n$, the expected value of the future is $X_n$ itself.

(b) Now use the additional fact that $\{X_n\}$ is a MC to deduce that $E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) = X_n$, $n \geq 0$: Given the present state, $X_n$, the expected value of the future is $X_n$ itself, independent of the past.

This is an example of a martingale, a stochastic process for which $E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) = X_n$, $n \geq 0$. It models a fair game in gambling: Let $X_n$ denote the total fortune after your $n^{th}$ gamble. Each consecutive gamble on average yields no profit and no loss given the current total earnings, independent of the past. The net earning on the $n^{th}$ gamble is the increment $\Delta_n = X_n - X_{n-1}$, and we can rewrite $X_n = X_0 + \Delta_1 + \cdots + \Delta_n$, $n \geq 0$. Then the Martingale property equivalently states that $E(\Delta_{n+1}|X_n, X_{n-1}, \ldots, X_0) = 0$, $n \geq 0$. Using the general fact that $E[E(X|Y)] = E(X)$, we see that $E(\Delta_n) = 0$, $n \geq 1$ and $E(X_n) = X_0$, $n \geq 0$. Thus a martingale $X_n$ is like a symmetric random walk (mean 0 increments), except the increments $\Delta_i$ need not be iid.

5. Consider a Markov chain with finite state space $S = \{1,2,\ldots, b\}$, and transition matrix $P$. Suppose that it is irreducible, and that each column of $P$ sums to 1. Prove that the limiting (stationary) distribution (e.g., the solution to $\pi = \pi P$) is the uniform distribution over the $b$ elements, $\pi_j = 1/b$, $j \in S$.

6. Chapter 4, Page 270, Exercises 41 and 42.