HMWK 5

Problems from the Text

Ch. 6, Page 409: 12, 13, 22

1. Chapter 6, Problem 12.

2. (a) If the state is the number of individuals at time \( t \), we get a B&G process with
\[
\lambda_n = n\lambda + \theta, \quad n < N,
\]
\[
\lambda_n = n\lambda, \quad n \geq N,
\]
\[
\mu_n = n\mu, \quad n \geq 0.
\]

(b) Let \( P_i \) be the limiting probability that the system is in state \( i \). We are looking for \( \sum_{i=3}^{\infty} P_i \). We have \( \lambda_k P_k = \mu_{k+1} P_{k+1} \).

This yields
\[
P_1 = \frac{\theta}{\mu} P_0
\]
\[
P_2 = \frac{\lambda + \theta}{2\mu} P_1 = \frac{\theta(\lambda + \theta)}{2\mu^2} P_0
\]
\[
P_3 = \frac{2\lambda + \theta}{2\mu} P_2 = \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{6\mu^3} P_0
\]
For \( k \geq 4 \), we get
\[
P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1},
\]
which implies
\[
P_k = \frac{(k-1)(k-2)\cdots 3}{k(k-1)\cdots 4} \left( \frac{\lambda}{\mu} \right)^{k-3} P_3 = \frac{3}{k} \left( \frac{\lambda}{\mu} \right)^{k-3} P_3;
\]
Therefore \( \sum_{k=3}^{\infty} P_k = 3 \left( \frac{\lambda}{\mu} \right)^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k} \left( \frac{\lambda}{\mu} \right)^k \).

But \( \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\lambda}{\mu} \right)^k = \log \left( \frac{1}{1 - \frac{\lambda}{\mu}} \right) \) if \( \frac{\lambda}{\mu} < 1 \).

So
\[
\sum_{k=3}^{\infty} P_k = 3 \left( \frac{\lambda}{\mu} \right)^3 P_3 \left[ \log \left( \frac{\mu}{\mu - \lambda} \right) - \frac{\lambda}{\mu} - \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^2 \right]
\]
\[
= 3 \left( \frac{\lambda}{\mu} \right)^3 \left[ \log \left( \frac{\mu}{\mu - \lambda} \right) - \frac{\lambda}{\mu} - \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^2 \right] \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{6\mu^3} P_0.
\]
Now \( \sum_{i=0}^{\infty} P_i = 1 \) implies
\[
P_0 = \left[ 1 + \frac{\theta}{\mu} + \frac{\theta(\lambda + \theta)}{2\mu^2} + \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{2\lambda^3} \right] \left[ \log \left( \frac{\mu}{\mu - \lambda} \right) - \frac{\lambda}{\mu} - \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^2 \right]^{-1}.
\]

And finally,
\[
\sum_{k=3}^{\infty} P_k = \left[ \left( \frac{1}{2\lambda^3} \right) \log \left( \frac{\mu}{\mu - \lambda} \right) - \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^2 \right] \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{1 + \theta + \frac{\theta(\lambda + \theta)}{2\mu^2} + \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{2\lambda^3} \left[ \log \left( \frac{\mu}{\mu - \lambda} \right) - \frac{\lambda}{\mu} - \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^2 \right]}.
\]

13.

\[
\lambda P_0 = \mu P_1
\]
\[
\lambda P_1 = \mu P_2.
\]

Solution: \( P_1 = \rho P_0 \), \( P_2 = \rho^2 P_0 \), where \( P_0 = \left( 1 + \rho + \rho^2 \right)^{-1} \).

(a)
\[
l = \sum_{n} n P_n = P_1 + 2P_2 = \frac{\rho + 2\rho^2}{1 + \rho + \rho^2}.
\]
(b) By PASTA, this is the long run proportion of time that \( X(t) \neq 2, = P_0 + P_1 = 28/37. \)

(c) Changing \( \mu = 4 \) to \( \mu = 8 \), so that \( \rho \) is changed from \( 3/4 \) to \( 3/8 \), \( P_0 + P_1 \) changes to \( 88/97 \), so the answer is \( (88/97) - (28/37) = 0.90 - 0.76 = 0.14 \).

Problems not from the textbook

1. According to a Poisson process at rate \( \lambda = 20 \) per day, a company buys units (100 share blocks) of stock A and holds on to each unit, independently of other units, for \( H \) days, where \( H \) has an exponential distribution with \( E(H) = 60 \) (days).

   Assume that initially (time \( t = 0 \)) no units of stock A are held.

   (a) Compute the expected number of units held at times \( t = 20, t = 50 \) and \( t = 90 \) days.

   **SOLUTION:** This is an example of a M/G/\( \infty \) queue, with “service time” distribution \( P(H > s) = e^{-\mu s} \) where \( \mu = 1/60 \). Let \( X(t) \) denote the number of units held at time \( t \). We know that \( X(t) \) has a Poisson distribution with mean \( \alpha(t) = \frac{\lambda}{\mu} \int_0^t P(H > s)ds = \rho(1 - e^{-\mu t}), \)

   where \( \rho = \frac{\lambda}{\mu} = 1200 \). Thus \( \alpha(t) = 1200(1 - e^{-t/60}) \).

   (b) What is the long-run average number of units held by the company?

   **SOLUTION:** “Long-run time-average, \( \lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds \), equals the mean of the limiting distribution”, \( \sum_j jP_j \), and here the limiting distribution is Poisson with mean \( \rho = \frac{\lambda}{\mu} = \frac{(20)(60)}{1200} = 1200 \).

   (c) Repeat (a), (b) when \( H \) has a uniform distribution on \( (40, 80) \).

   **SOLUTION:**

\[
P(H > s) = \begin{cases} 
1, & \text{if } s \in [0, 40); \\
\frac{80-s}{40}, & \text{if } s \in [40, 80); \\
0, & \text{if } s \geq 80.
\end{cases}
\]

So: In this case we must pay attention to the fact that the tail \( P(H > t) = 1, t \in [0, 40) \), so that

\[
\alpha(t) = \lambda \int_0^t ds = \lambda t, \ t \in [0, 40),
\]

and then for any \( t \in [40, 80) \)

\[
\alpha(t) = 40\lambda + \lambda \int_4^t \frac{80-s}{40} ds,
\]

and finally, \( \alpha(t) = \rho, \ t \geq 80. \)

2. (a) In addition to a remaining service time \( S_r \sim exp(\mu) \) and a remaining interarrival time \( T \sim exp(\lambda) \) we now also have a remaining time until the next jam \( J \sim exp(\gamma) \). These are independent and independent of the past, so similar to the M/M/1 queue, \( X(t) \) satisfies the Markov property. It is birth and death because the state only changes by magnitude 1. Note that the birth rate is \( \lambda_i = \lambda, i \geq 0 \), and the death rates are \( \mu_i = \mu + \gamma, i \geq 1 (\mu_0 = 0) \).
i. \( \lambda P_n = (\mu + \gamma)P_{n+1}, \ n \geq 0 \). Setting \( \overline{\mu} = \mu + \gamma \), we see that this is the same set of equations as for a regular M/M/1 queue with arrival rate \( \lambda \) and service rate \( \overline{\mu} \).

Thus, stability if and only if \( \overline{\rho} = \lambda / \overline{\mu} < 1 \), and solution \( P_n = (1 - \rho)^n \), \( n \geq 0 \).

In effect, this is an M/M/1 queue in which service times are iid exponential with rate \( \overline{\mu} \): Each customer remains in service for an iid amount of time \( S = \min\{S, J\} \sim \text{exp}(\mu + \gamma) \).

ii. \( 1 - P_0 = \rho \).

iii. \( l = \sum_{n=0}^{\infty} nP_n = \frac{\overline{\rho}}{1 - \overline{\rho}} \).

iv. Each job when in service will be removed independent of all others with probability \( P(J < S) = \gamma / (\gamma + \mu) \). Thus this probability is the proportion that get removed.

v. A. PASTA implies that this is equal to the proportion of time that the system is empty/busy; \( P_0 = 1 - \rho \) and \( 1 - P_0 = \rho \).

B. A jam time removes a job if and only if the system is busy when a jam occurs. Thus we want the proportion of jam times that find the system busy, which by PASTA is equal to the proportion of time that the system is busy, \( \rho \). The point here is that we can treat the jam times as if they are “arrivals” and apply PASTA.

3. Consider an irreducible discrete-time Markov chain \( \{X_n\} \) with finite state space \( S = \{1, 2, \ldots N\} \), and transition matrix \( P \). Consider the set of \( N \) equations, \( \pi = \pi P \), and the additional equation \( \pi_1 + \cdots + \pi_N = 1 \). Thus a total of \( N + 1 \) equations. (Because of the irreducibility and finite state space, we know that there is a unique probability solution.)

Show that if you remove any one of the first \( N \) equations, then it can be generated from the remaining \( N \) anyhow, thus it is redundant.

**SOLUTION:** Without loss of generality, let us suppose that it is the first equation \( \pi_1 = \sum_{i=1}^{N} \pi_i P_{i,1} \) that we wish to replace with \( \sum_{j=1}^{N} \pi_j = 1 \). It then suffices to show that the remaining \( N - 1 \) equations can be combined to reproduce equation 1.

To this end, we simply sum them up yielding

\[
\pi_2 + \cdots + \pi_N = \pi_1(\sum_{j=2}^{N} P_{1,j}) + \cdots + \pi_N(\sum_{j=2}^{N} P_{N,j}) = \pi_1(1 - P_{1,1}) + \cdots + \pi_N(1 - P_{N,1}).
\]

Now using \( \sum_{j=1}^{N} \pi_j = 1 \) on both sides yields

\[
1 - \pi_1 = 1 - \sum_{i=1}^{N} \pi_i P_{i,1},
\]

or

\[
\pi_1 = \sum_{i=1}^{N} \pi_i P_{i,1},
\]

as was to be shown.