### HMWK 7 Solutions

1. $X_n = Y_1 \times Y_2 \times \cdots \times Y_n$ where $X_0 = 1$ and the $Y_i$ are iid with $P(Y = 0.5) = P(Y = 1) = P(Y = 1.5) = 1/3$. Argue that $X_n$ converges and find the limit.

**SOLUTION:** Since $E(Y) = 1$, $X_n$ is a non-negative martingale, so it must converge $w.p.1$ to a rv $X$ such that $E(X) < \infty$ by the martingale convergence theorem. Because $R_n = \ln(X_n) = \sum_{i=1}^n \ln(Y_n)$ is a random walk with iid increments $\ln(Y_n)$, we conclude by SLLN that $\lim_{n} \frac{R_n}{n} = E[\ln(Y)] = \frac{1}{3} \ln \frac{3}{2} < 0$. So the random walk has negative drift; $\lim_{n} R_n = -\infty$, $w.p.1$. Thus $X_n = e^{R_n} \to 0$, $w.p.1$.

2. Consider the Gambler’s ruin Markov chain $\{X_n\}$ on $\{0, 1, 2, 3, 4\}$ ($N = 4$) with transition matrix

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
0 & 1/2 & 0 & 1/2 & 0 \\
0 & 0 & 1/2 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Suppose that $X_0 = 1$. Does $X_n$ converge $w.p.1$? Find the limiting rv $X$ if so. Repeat for each initial condition $X_0 = i$, $i = 0, 1, 2, 3, 4$.

**SOLUTION:** First we can easily check that $\{X_n\}$ is a non-negative martingale for each initial state $X_0 = i$, $i \in \{0, 1, 2, 3, 4\}$. This is because $X_n$ is the simple symmetric random walk (already a known MG) away from the boundaries (e.g., when $X_0 = i \in \{1, 2, 3\}$), and at a boundary (e.g., $X_0 = 0$ or $X_0 = 4$) it remains constant. Thus $X_n$ converges $w.p.1$ to a rv $X$ with $E(X) < \infty$. From the gambler’s ruin problem we know that states $0$ and $4$ are absorbing states to which $X_n$ eventually hits, and the probability it hits $N = 4$ before $0$ when $X_0 = i$ is given by $P_i = \frac{i}{4}$, and $1 - P_i = 1 - \frac{i}{4}$ is the probability that $0$ is hit first (ruin). Thus for $X_0 = i$, the limiting rv $X = X(i)$ has distribution $P(X = 0) = 1 - \frac{i}{4}$, $P(X = 4) = \frac{i}{4}$.

(Using stopping time $\tau(i) = \min \{n \geq 0 : X_n \in \{N, 0\} \mid X_0 = i\}$, (the first time the gambler stops), we have $w.p.1$, that if $X_0 = i$, then $X_n \to X = X_{\tau(i)}$.)

The point here is that for each fixed $i$, the chain started with $X_0 = i$ is a MG to which the convergence theorem applies; it has its own limiting rv for each $i$. (Of course If $X_0 = 0$, then $X_n = 0$, $n \geq 1$, and so $X = 0$ $w.p.1$. Similarly if $X_0 = 4$, then $X_n = 4$, $n \geq 1$, and so $X = 4$ $w.p.1$.)

3. **Continuation:** Consider the Markov chain $\{X_n\}$ on $\{0, 1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 0 & 1/3 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 1/2 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
5. Consider the BLM, with $S_0 = 1$ and $S_n = Y_1 \cdots Y_n$, $n \geq 1$ where $d = 0.5$, $u = 1.5$ and $p = 0.50$.

(a) Show that $E(S_n) = 1$, $n \geq 0$, but wp1, $S_n \to 0$ as $n \to \infty$.

**SOLUTION:** This is the same analysis as Problem 1. $E\ln(Y) = (1/2)(\ln(3/2) + \ln(1/2)) = (1/2)\ln(3/4) < 0$.

(b) Let $\epsilon > 0$ be very small. Let us change $u$ to be $(1.5)(1+\epsilon)$ so that it is strictly larger than 1.5. We keep all other parameters the same as before. Show now that $E(S_n) \to \infty$, and that if $\epsilon$ is chosen small enough it still remains true that $S_n \to 0$. On average you will become infinitely rich, but with certainty you will go broke! (Interesting?)

**SOLUTION:** $E(Y) = (1/2)(3/2 + 3/2(\epsilon) + 1/2) = 1 + (3/4)\epsilon > 1$. Thus $E(S_n) = E^n(Y) \uparrow \infty$. 

4. At time 0 an urn contains one red ball and one blue ball. At each time $n$ after $(n = 1, 2, 3, \ldots)$ a ball is chosen at random from the urn and put back in together with one more ball of the same color. (So at time $n$ there are $n + 2$ balls in the urn.) Let $X_n$ denote the proportion of balls at time $n$ that are red. Show that \{X_n\} is a MG. Does it converge?

**SOLUTION:** $X_0 = 1/2$. So at time $n = 1$, either $X_1 = 1/3$, wp $= 1/2$, or $X_1 = 2/3$, wp $= 1/2$; thus $E(X_1|X_0 = 1/2) = (1/3 + 2/3)(1/2) = 1/2 = X_0$. For general $n$: We note that \{X_n\} is a Markov chain (but with transition probabilities that depend on time $n$): Given $X_n$, $X_{n+1}$ is either $(n + 2)X_n + 1)/(n + 3)$ or $(n + 2)X_n)/(n + 3)$ with probabilities $X_n$ and $1 - X_n$ respectively independent of the past \{X_0, \ldots, X_{n-1}\}.

Thus we get the MG property via

$$E(X_{n+1} | X_n, \ldots, X_0) = E(X_{n+1} | X_n) \text{ (Markov property)}$$

$$= [(n + 2)X_n + 1)/(n + 3)]X_n + [(n + 2)X_n)/(n + 3)](1 - X_n)$$

$$= (n + 3)^{-1}[(n + 2)X_n + 1)X_n + (n + 2)X_n)(1 - X_n)]$$

$$= (n + 3)^{-1}(n + 3)X_n$$

$$= X_n.$$ 

Does $X_n$ converge wp1?

Find the limiting rv $X$ if so (it depends on initial conditions).

**SOLUTION:** The point here is that just as for the gambler’s ruin problem above, \{X_n\} is a non-negative MG for each initial state $X_0 = i$, $i \in \{0, 1, 2, 3, 4\}$. (The $i^{th}$ row of $P$ has expected value $i$; $\sum jP_{i,j} = i$; $E(X_n+1|X_n = i) = E(X_{n+1} | X_n = i)$ (Markov property) = $\sum jP_{i,j} = i$).

Thus $X_n$ converges wp1 to a rv $X$ with $E(X) < \infty$. Moreover using stopping time $\tau(i) = \min\{n \geq 0 : X_n \in \{N, 0\}| X_0 = i\}$, the stopped MG $\overline{X}_n = X_{n\wedge \tau} \leq 4$ is bounded hence UI, so given $X_0 = i$, the optional stopping theorem yields $E(X_\tau) = E(X_0) = i$, or $p_iN + (1 - p_i)0 = i$ yielding $P_i = 1 \frac{1}{4}$, and $1 - P_i = 1 - 1 \frac{1}{4}$ exactly as for the gambler’s ruin problem. Thus for $X_0 = i$, the limiting rv $X = X(i)$ has distribution $P(X = 0) = 1 - \frac{i}{4}$, $P(X = 4) = \frac{i}{4}$, exactly as for the gambler’s ruin problem.
\[ E \ln(Y) = \frac{1}{2}(\ln(3/2)+\ln(1/2)+\ln(1+\epsilon)) = \frac{1}{2}[\ln(3/4)+\ln(1+\epsilon)] < 0 \text{ as long as } \epsilon \text{ is sufficiently small. To be precise, we need } (3/4)(1+\epsilon) < 1, \text{ or equivalently } 1+\epsilon < (4/3), \text{ yielding } \epsilon < 1/3. \epsilon = 1/6, \text{ for example, will work.} \]

(c) **Continuation:** Under the conditions in (b), show that \( \{S_n\} \) while not a MG is in fact a SUBMG.

**SOLUTION:** This is because now \( E(Y) > 1 \). Using the recursion \( S_{n+1} = S_nY_{n+1} \), we already know that \( S_n \) forms a MC, and we obtain the SUBMG property via \( E(S_{n+1} | \mathcal{F}_n) = S_nE(Y_{n+1} | \mathcal{F}_n) = S_nE(Y_{n+1}) > S_n \).