1. Consider a Poisson process at rate $\lambda$, and let $N(t)$ denote the number of arrivals (points) during the time interval $[0, t]$.

Let $X(t) = N(t) - \lambda t$, $t \geq 0$. Show that $\{X(t)\}$ is a mean 0 martingale in continuous time: For all $t, h \geq 0$, $E(X(t+h) | X(t), \{X(u) : 0 \leq u < t\}) = X(t)$. Now also show that $M(t) \stackrel{\text{def}}{=} X^2(t) - \lambda t$ is also a mean 0 MG.

2. The price of a commodity moves according to a BM, $X(t) = \sigma B(t) + \mu t$, with variance term $\sigma^2 = 4$ and drift $\mu = 3$.

   (a) Given that the price is 10 at time $t = 3$, what is the probability that the price is above 12 at time $t = 5$?
   
   (b) Given that the price starts at 1 at time $t = 0$, what is the probability it will hit 2 before hitting level 0?
   
   (c) Given that the price starts at 10, what is the probability it will ever reach a low of 3?
   
   (d) Given that the price starts at 10 at time $t = 0$, what is the expected amount of time until it hits 15?

3. If $X_1(t) = \sigma_1 B_1(t) + \mu_1 t$ and $X_2(t) = \sigma_2 B_2(t) + \mu_2 t$ are independent BM’s, then argue that $X(t) \stackrel{\text{def}}{=} X_1(t) - X_2(t)$ is also a BM with $\sigma = ?$ and $\mu = ?$

4. Continuation: If $X_1(t) = 2B_1(t) + 5t$ with $X_1(0) = 4$ is a BM, and independently $X_2(t) = 3B_2(t) + 3t$ with $X_2(0) = 0$ is another BM, compute the probability that in the future the two processes meet (e.g., that eventually $X_1(t) = X_2(t)$ for some $t$.)

5. If $\{N(t)\}$ is a Poisson counting process and independently $\{B(t)\}$ is a Brownian motion, then does the sum $Z(t) = N(t) + B(t)$ have stationary increments? Independent increments?

6. Suppose that $Z$ is a rv with a $N(0, 1)$ distribution. Define a stochastic process via $X(t) = \sqrt{t}Z$, and note that for each $t > 0$, $X(t)$ has a $N(0, t)$ distribution. Is this Brownian motion?