IEOR 4106, Midterm Exam, Spring 2018. 75 Minutes. 100 Points Total. Professor K. Sigman

Open Notes (anything on the course website plus your notes from class), but no books and no electronic devices of any kind.

Make sure to show/justify your work, don't just write down an answer with no explanation!

1. (30 points, 10 each) A gambler George starts with $i = 1$ (dollar), and plays according to the Gamber's ruin problem, with $p = 2/3$, but with N a random variable: $P(N = 2) = P(N = 3) = 1/2$. The idea is that just before George starts, he first flips a fair coin (once) to decide the value of N (Heads = 2, Tails = 3). Then he plays until reaching that value of N or going broke, whichever happens first, then he stops and goes home. He starts with $i = 1$ (dollar).

   (a) Given that $N = 2$, what is the probability that the gambler goes home broke? (Exact numerical answer must be given.)

   SOLUTION: Probability = 1/3 as is clear: When $N = 2$ with $i = 1$, the game is over after the first gamble (either $1 \to 2$ or $1 \to 0$), and with probability $q = 1/3$ that gamble will end the game with the gambler broke ($1 \to 0$).

   Formally (but not needed here for credit, the above argument is fine):
   We want $1 - P_i(N)$, where
   $$P_i(N) = \frac{1 - (q/p)^i}{1 - (q/p)^N}.$$
   And we would use $i = 1, q/p = 1/2, N = 2$:
   $$P_1(2)) = \frac{1 - (1/2)}{1 - (1/2)^2} = 2/3;$$
   $1 - P_1(2) = 1/3$.
   If $N = 3$ (we need this computation for (b) below), then we do need to use the more general formula given above and we want
   $$P_1(3) = \frac{1 - (1/2)}{1 - (1/2)^3} = 4/7;$$
   Answer $= 1 - P_1(3) = 3/7$.

   (b) Compute the probability (exact numerical answer) that the gambler will go home broke.

   SOLUTION: By conditioning on $N = 2$ and $N = 3$ (using what we computed in (a) for both cases), our answer will be a $50 - 50$ weighted average of using $N = 2, 3$ because each case will ocurr with probability 1/2 by the fair coin assumption:
   $(1/2)(1 - P_1(2))) + (1/2)(1 - P_1(3))) = (1/2)(1/3 + 3/7) = 8/21$

   (c) Explain (but you do not need to carry out the computation) how to compute the probability that George will go home after at most ($\leq$) 7 gambles.

   SOLUTION:

   If $N = 2$, then the game is over in 1 gamble with certainty, hence over after at most 7 gambles with certainty. But if $N = 3$, then we must consider the transition matrix P for the Gambler's ruin problem Markov chain on $\{0, 1, 2, 3\}$, and compute $P^{(7)} = P^7$ and use as our answer $P^{(7)}_{1,0} + P^{(7)}_{1,3}$; both 0 and 3 are absorbing states. When $N = 3$, P is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 1. \end{pmatrix}$$

Then one must compute P^7 (not required here on the exam; just let $P^7 = (P^{(7)}_{i,j})$) and express the answer in terms of this.) Thus our answer is the $50-50$ weighted average:
$(1/2)(1 + (P^{(7)}_{1,0} + P^{(7)}_{1,3}))$

2. (10 points) A certain stochastic process $\{X_n : n \geq 0\}$ is believed by a researcher to be a Markov chain with state space $\mathcal{S} = \{1, 2\}$ and transition matrix of the form

$$P = \begin{pmatrix} 0.5 & 0.5 \\ p & 1-p \end{pmatrix},$$

for some $0 < p < 1$ unknown. By looking at the values $\{X_0, \ldots, X_n\}$ for a very very large time n, the researcher estimated that the process visits state 1 approximately 40% of the time, and visits state 2 approximately 60% of the time. From this, give a very reasonable choice of what the numerical value of p should be.

SOLUTION:

For any $0 < p < 1$, the chain is irreducible and since it has a finite state space it must have a unique limiting distribution $\pi = (\pi_1, \pi_2)$ satisfying $\pi = \pi P$. From the information they told us: We can assume that (as a very precise approximation) $\pi_1 = 0.40$ and $\pi_2 = 0.60$, since indeed the π_i are (by definition) the long run proportions of time the chain visits states 1 and 2 respectively. Then we plug that into $\pi = \pi P$ to solve for p:

$$\pi_1 = (0.5)\pi_1 + p\pi_2 \quad (1)$$
$$\pi_2 = (0.5)\pi_1 + (1-p)\pi_2. \quad (2)$$

The first equation becomes $0.40 = (0.5)(0.40) + p(0.60)$, from which we easily solve; $p = 1/3$. (The second equation yields the same solution if you wish to use that instead.)

3. (20 points, 10 each)

Let $\psi = \{t_n : n \geq 1\}$ be a Poisson process at rate λ, with counting process $\{N(t) : t \geq 0\}$. For a fixed $t > 0$, let $T = N(t) + 1$.

(a) Compute $E(T)$

SOLUTION:

$N(t)$ is Poisson distributed with mean λt , so $E(T) = E(N(t)) + 1 = \lambda t + 1$.

(b) Compute $E(t_T)$; the expected time that the T^{th} point occurs.

SOLUTION:

There are 2 Methods:

METHOD I (Wald's Equation): Letting $\{X_i\}$ denote the iid interarrival times; they are distributed as exponential with rate λ (hence $E(X) = 1/\lambda$). $t_n = X_1 + \cdots + X_n = \sum_{i=1}^n X_i$, and hence

$$t_T = \sum_{i=1}^{T} X_i,$$

$t_T = t_{N(t)} + 1$ is the *first point strictly after time* t; $T = N(t) + 1 = \min\{n \geq 1 : t_n > t\}$ is a stopping time with respect to $\{t_n : n \geq 1\}$, equivalently with respect to $\{X_n : n \geq 1\}$, and $E(T) = \lambda t + 1 < \infty$, and $E(X) = 1/\lambda < \infty$. Thus by Wald's equation we have $E(t_T) = E(T)E(X) = (\lambda t + 1)/\lambda = t + 1/\lambda$.

NOTE: $N(t)$ is *not* a stopping time; $N(t) = \max\{n : t_n \leq t\}$.

$t_{N(t)}$ is the *last point before or at time* t:

$t_{N(t)} \leq t < t_{N(t)+1}$. But $N(t) + 1$ is a stopping time.

METHOD II: Memoryless property of the exponential distribution:

$t_T = t_{N(t)+1} = t + A(t)$, where $A(t) = t_{N(t+1)} - t$ is the *remaining* interarrival time from time t onwards. By the memoryless property of the exponential distribution, $A(t)$ is exponential at rate λ (and independent of the past), hence $E(t_T) = t + E(A(t) = t + 1/\lambda$.

4. (40 points, 10 each) Consider an $M/G/\infty$ queue, with arrival rate $\lambda = 2$ and iid service times distributed as (CDF) $G(x) = P(S \leq x) = 1 - \frac{1}{(1+x)^2}$, $x \geq 0$, but at time $t = 0$, two initial customers $C_0(1), C_0(2)$ enter service with iid independent service times Y_1, Y_2 distributed as exponential at rate 1.

   (a) Let $X(t),\ X(0) = 0$ denote the number of busy servers at time t *not including the 2 initial customers*. Compute $E(X(1))$.

   SOLUTION:

   $X(t)$ has a Poisson distribution with mean

   $$\alpha(t) = \lambda \int_0^t P(S > u)du = 2 \int_0^t \frac{1}{(1+u)^2} du = 2(1 - \frac{1}{1+t}).$$

   $E(X(1)) = \alpha(1) = 1$.
   Also, $E(S) = \int_0^\infty \frac{1}{(1+u)^2} du = 1$; $\rho = \lambda E(S) = 2(1) = 2$.

   (b) *Continuation:*

   Compute $E(X(\infty)) = \lim_{t\to\infty} E(X(t))$.

   SOLUTION: $\lim_{t\to\infty} E(X(t)) = \lim_{t\to\infty} \alpha(t) = \rho = \lambda E(S) = \alpha(\infty) = 2(1) = 2$.

   (c) Let $Z(t),\ Z(0) = 2$ denote the number of busy servers at time t *including the 2 initial customers*. Compute $E(Z(1))$.

   SOLUTION: The 2 initial customers will depart at times Y_1 and Y_2 respectively; hence each is still in the system at time t if and only if $Y_1 > t$ and $Y_2 > t$ respectively. Thus

   $Z(t) = X(t) + I\{Y_1 > t\} + I\{Y_2 > t\}$, and $E(I\{Y_1 > t\}) = E(I\{Y_2 > t\}) = P(Y_1 > t) = e^{-t}$.
   Thus
   $E(Z(t)) = E(X(t)) + 2P(Y_1 > t) = \alpha(t) + 2e^{-t}$.
   $E(Z(1)) = \alpha(1) + 2e^{-1} = 1 + 2e^{-1}$.

   (d) *Continuation:* Compute $E(Z(\infty)) = \lim_{t\to\infty} E(Z(t))$.

   SOLUTION: From (c), $E(Z(t)) = \alpha(t) + 2e^{-t}$, and $e^{-t} \to 0$ as $t \to \infty$ hence the answer is the same as in (b); $\rho = 2$.