1. (30 points, 10 each) A gambler George starts with $i = 1$ (dollar), and plays according to the Gambler’s ruin problem, with $p = 2/3$, but with $N$ a random variable: $P(N = 2) = P(N = 3) = 1/2$. The idea is that just before George starts, he first flips a fair coin (once) to decide the value of $N$ (Heads = 2, Tails = 3). Then he plays until reaching that value of $N$ or going broke, whichever happens first, then he stops and goes home. He starts with $i = 1$ (dollar).

   (a) Given that $N = 2$, what is the probability that the gambler goes home broke? (Exact numerical answer must be given.)

   **SOLUTION:** Probability = $1/3$ as is clear: When $N = 2$ with $i = 1$, the game is over after the first gamble (either $1 \rightarrow 2$ or $1 \rightarrow 0$), and with probability $q = 1/3$ that gamble will end the game with the gambler broke ($1 \rightarrow 0$).

   Formally (but not needed here for credit, the above argument is fine):

   We want $1 - P_i(N)$, where

   $$P_i(N) = \frac{1 - (q/p)^i}{1 - (q/p)^N}.$$  

   And we would use $i = 1, q/p = 1/2, N = 2$:

   $$P_i(2)) = \frac{1 - (1/2)}{1 - (1/2)^2} = 2/3;$$

   $1 - P_i(2) = 1/3$.

   If $N = 3$ (we need this computation for (b) below), then we do need to use the more general formula given above and we want

   $$P_i(3) = \frac{1 - (1/2)}{1 - (1/2)^3} = 4/7;$$

   Answer = $1 - P_i(3) = 3/7$.

   (b) Compute the probability (exact numerical answer) that the gambler will go home broke.

   **SOLUTION:** By conditioning on $N = 2$ and $N = 3$ (using what we computed in (a) for both cases), our answer will be a 50–50 weighted average of using $N = 2, 3$ because each case will occur with probability 1/2 by the fair coin assumption:

   $$(1/2)(1 - P_i(2))) + (1/2)(1 - P_i(3))) = (1/2)(1/3 + 3/7) = 8/21$$

   (c) Explain (but you do not need to carry out the computation) how to compute the probability that George will go home after at most ($\leq$) 7 gambles.

   **SOLUTION:**

   If $N = 2$, then the game is over in 1 gamble with certainty, hence over after at most 7 gambles with certainty. But if $N = 3$, then we must consider the transition matrix $P$ for the Gambler’s ruin problem Markov chain on $\{0, 1, 2, 3\}$, and compute $P^{(7)} = P^7$ and use as our answer $P^{(7)}_{1,0} + P^{(7)}_{1,3}$; both 0 and 3 are absorbing states. When $N = 3$, $P$ is given by
\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1/3 & 0 & 2/3 & 0 \\
0 & 1/3 & 0 & 2/3 \\
0 & 0 & 0 & 1.
\end{pmatrix}
\]

Then one must compute \(P^7\) (not required here on the exam; just let \(P^7 = (P_{i,j})\)) and express the answer in terms of this.) Thus our answer is the 50 – 50 weighted average:

\[
(1/2)(1 + (P_{1,0}^{(7)} + P_{1,3}^{(7)}))
\]

2. (10 points) A certain stochastic process \(\{X_n : n \geq 0\}\) is believed by a researcher to be a Markov chain with state space \(S = \{1, 2\}\) and transition matrix of the form

\[
P = \begin{pmatrix}
0.5 & 0.5 \\
p & 1 - p
\end{pmatrix},
\]

for some \(0 < p < 1\) unknown. By looking at the values \(\{X_0, \ldots, X_n\}\) for a very very large time \(n\), the researcher estimated that the process visits state 1 approximately 40% of the time, and visits state 2 approximately 60% of the time. From this, give a very reasonable choice of what the numerical value of \(p\) should be.

**SOLUTION:**

For any \(0 < p < 1\), the chain is irreducible and since it has a finite state space it must have a unique limiting distribution \(\pi = (\pi_1, \pi_2)\) satisfying \(\pi = \pi P\). From the information they told us: We can assume that (as a very precise approximation) \(\pi_1 = 0.40\) and \(\pi_2 = 0.60\), since indeed the \(\pi_i\) are (by definition) the long run proportions of time the chain visits states 1 and 2 respectively. Then we plug that into \(\pi = \pi P\) to solve for \(p\):

\[
\begin{align*}
\pi_1 &= (0.5)\pi_1 + p\pi_2 \quad (1) \\
\pi_2 &= (0.5)\pi_1 + (1 - p)\pi_2. \quad (2)
\end{align*}
\]

The first equation becomes \(0.40 = (0.5)(0.40) + p(0.60)\), from which we easily solve; \(p = 1/3\). (The second equation yields the same solution if you wish to use that instead.)

3. (20 points, 10 each)

Let \(\psi = \{t_n : n \geq 1\}\) be a Poisson process at rate \(\lambda\), with counting process \(\{N(t) : t \geq 0\}\). For a fixed \(t > 0\), let \(T = N(t) + 1\).

(a) Compute \(E(T)\)

**SOLUTION:**

\(N(t)\) is Poisson distributed with mean \(\lambda t\), so \(E(T) = E(N(t)) + 1 = \lambda t + 1\).

(b) Compute \(E(t_T)\); the expected time that the \(T^{th}\) point occurs.

**SOLUTION:**

There are 2 Methods:

**METHOD I (Wald’s Equation):** Letting \(\{X_i\}\) denote the iid interarrival times; they are distributed as exponential with rate \(\lambda\) (hence \(E(X) = 1/\lambda\)). \(t_n = X_1 + \cdots + X_n = \sum_{i=1}^{n} X_i\), and hence

\[
t_T = \sum_{i=1}^{T} X_i,
\]
\( t_T = t_{N(t)} + 1 \) is the first point strictly after time \( t \); \( T = N(t) + 1 = \min \{ n \geq 1 : t_n > t \} \) is a stopping time with respect to \( \{ t_n : n \geq 1 \} \), equivalently with respect to \( \{ X_n : n \geq 1 \} \), and \( E(T) = \lambda t + 1 < \infty \), and \( E(X) = 1/\lambda < \infty \). Thus by Wald’s equation we have \( E(t_T) = E(T)E(X) = (\lambda t + 1)/\lambda = t + 1/\lambda \).

**NOTE:** \( N(t) \) is not a stopping time; \( N(t) = \max \{ n : t_n \leq t \} \).

\( t_{N(t)} \) is the last point before or at time \( t \):

\( t_{N(t)} \leq t < t_{N(t)+1} \). But \( N(t) + 1 \) is a stopping time.

**METHOD II: Memoryless property of the exponential distribution:**

\( t_T = t_{N(t)+1} = t + A(t) \), where \( A(t) = t_{N(t)+1} - t \) is the remaining interarrival time from time \( t \) onwards. By the memoryless property of the exponential distribution, \( A(t) \) is exponential at rate \( \lambda \) (and independent of the past), hence \( E(t_T) = t + E(A(t)) = t + 1/\lambda \).

4. (40 points, 10 each) Consider an \( M/G/C \) distributed as (CDF)

\( \text{Continuation:} \)

(a) Let \( Z_0 \) is a stopping time with respect to \( \{ X_n : n \geq 1 \} \), and \( E(\{ z \}) = 1/\lambda < \infty \). Thus by Wald’s equation we have \( E(t_T) = E(T)E(X) = (\lambda t + 1)/\lambda = t + 1/\lambda \).

**SOLUTION:**

The 2 initial customers will depart at times \( E(\{ z \}) = 1/\lambda < \infty \). Thus by Wald’s equation we have \( E(t_T) = E(T)E(X) = (\lambda t + 1)/\lambda = t + 1/\lambda \).

**Continuation:**

(b) \( t_{N(t)} \) is the last point before or at time \( t \):

\( t_{N(t)} \leq t < t_{N(t)+1} \). But \( N(t) + 1 \) is a stopping time.

**METHOD II: Memoryless property of the exponential distribution:**

\( t_T = t_{N(t)+1} = t + A(t) \), where \( A(t) = t_{N(t)+1} - t \) is the remaining interarrival time from time \( t \) onwards. By the memoryless property of the exponential distribution, \( A(t) \) is exponential at rate \( \lambda \) (and independent of the past), hence \( E(t_T) = t + E(A(t)) = t + 1/\lambda \).

4. (40 points, 10 each) Consider an \( M/G/\infty \) queue, with arrival rate \( \lambda = 2 \) and iid service times distributed as (CDF) \( G(x) = P(S \leq x) = 1 - \frac{1}{2} e^{-x} \), \( x \geq 0 \), but at time \( t = 0 \), two initial customers \( C_0(1), C_0(2) \) enter service with iid independent service times \( Y_1, Y_2 \) distributed as exponential at rate 1.

(a) Let \( X(t), X(0) = 0 \) denote the number of busy servers at time \( t \) not including the 2 initial customers. Compute \( E(X(1)) \).

**SOLUTION:**

\( X(t) \) has a Poisson distribution with mean

\[
\alpha(t) = \lambda \int_0^t P(S > u)du = 2 \int_0^t \frac{1}{1+u}du = 2(1 - \frac{1}{1+t}).
\]

\( E(X(1)) = \alpha(1) = 1 \).

Also, \( E(S) = \int_0^\infty \frac{1}{1+u}du = 1; \rho = \lambda E(S) = 2(1) = 2 \).

(b) \( \text{Continuation:} \)

Compute \( E(X(\infty)) = \lim_{t \to \infty} E(X(t)) \).

**SOLUTION:** \( \lim_{t \to \infty} E(X(t)) = \lim_{t \to \infty} \alpha(t) = \rho = \lambda E(S) = \alpha(\infty) = 2(1) = 2 \).

(c) Let \( Z(t), Z(0) = 2 \) denote the number of busy servers at time \( t \) including the 2 initial customers. Compute \( E(Z(1)) \).

**SOLUTION:** The 2 initial customers will depart at times \( Y_1 \) and \( Y_2 \) respectively; hence each is still in the system at time \( t \) if and only if \( Y_1 > t \) and \( Y_2 > t \) respectively. Thus

\[
\alpha(t) + 2e^{-t} = 1 + 2e^{-t}.
\]

(d) \( \text{Continuation:} \) Compute \( E(Z(\infty)) = \lim_{t \to \infty} E(Z(t)) \).

**SOLUTION:** From (c), \( E(Z(t)) = \alpha(t) + 2e^{-t}, \) and \( e^{-t} \to 0 \) as \( t \to \infty \) hence the answer is the same as in (b); \( \rho = 2 \).