IEOR 4106, Midterm Exam, Spring 2018. 75 Minutes. 100 Points Total. Professor K. Sigman

Open Notes (anything on the course website plus your notes from class), but no books and no electronic devices of any kind.

Make sure to show/justify your work, don't just write down an answer with no explanation!

1. (30 points, 10 each) A gambler George starts with $i = 1$ (dollar), and plays according to the Gamber's ruin problem, with $p = 2/3$, but with N a random variable: $P(N = 2) = P(N = 3) = 1/2$. The idea is that just before George starts, he first flips a fair coin (once) to decide the value of N (Heads = 2, Tails = 3). Then he plays until reaching that value of N or going broke, whichever happens first, then he stops and goes home. He starts with $i = 1$ (dollar).

   (a) Given that $N = 2$, what is the probability that the gambler goes home broke? (Exact numerical answer must be given.)

   (b) Compute the probability (exact numerical answer) that the gambler will go home broke.

   (c) Explain (but you do not need to carry out the computation) how to compute the probability that George will go home after at most ($\leq$) 7 gambles.

2. (10 points) A certain stochastic process $\{X_n : n \geq 0\}$ is believed by a researcher to be a Markov chain with state space $\mathcal{S} = \{1, 2\}$ and transition matrix of the form

$$P = \begin{pmatrix} 0.5 & 0.5 \\ p & 1-p \end{pmatrix},$$

   for some $0 < p < 1$ unknown. By looking at the values $\{X_0, \ldots, X_n\}$ for a very very large time n, the researcher estimated that the process visits state 1 approximately 40% of the time, and visits state 2 approximately 60% of the time. From this, give a very reasonable choice of what the numerical value of p should be.

3. (20 points, 10 each)

   Let $\psi = \{t_n : n \geq 1\}$ be a Poisson process at rate λ, with counting process $\{N(t) : t \geq 0\}$. For a fixed $t > 0$, let $T = N(t) + 1$.

   (a) Compute $E(T)$

   (b) Compute $E(t_T)$; the expected time that the T^{th} point occurs.

4. (40 points, 10 each) Consider an $M/G/\infty$ queue, with arrival rate $\lambda = 2$ and iid service times distributed as (CDF) $G(x) = P(S \leq x) = 1 - \frac{1}{(1+x)^2}$, $x \geq 0$, but at time $t = 0$, two initial customers $C_0(1), C_0(2)$ enter service with iid independent service times Y_1, Y_2 distributed as exponential at rate 1.

   (a) Let $X(t)$, $X(0) = 0$ denote the number of busy servers at time t *not including the 2 initial customers*. Compute $E(X(1))$.

   (b) *Continuation:* Compute $E(X(\infty)) = \lim_{t\to\infty} E(X(t))$.

   (c) Let $Z(t)$, $Z(0) = 2$ denote the number of busy servers at time t *including the 2 initial customers*. Compute $E(Z(1))$.

   (d) *Continuation:* Compute $E(Z(\infty)) = \lim_{t\to\infty} E(Z(t))$.