IEOR 4106, HMWK 1, Professor Sigman

1. An asset price starts off initially at price \$3.00 at the end of a day (day 0), and at the end of each consecutive day, independent of the past, the price goes up by one dollar (with probability $p = 0.7$) or down by one dollar (with probability $q = 0.3$).

   (a) What is the probability that the stock will reach \$11.00 before going down to 0?
   SOLUTION: Gambler's Ruin Problem: With $q/p = 3/7$; $P_3(11) = \frac{1-(q/p)^3}{1-(q/p)^{11}} = 0.921$.

   (b) What is the probability that the stock will reach \$10.00 before going down to a low of \$2.00?
   SOLUTION: Starting initially at 3 and going "up by 7" (to hit 10) before "down by 1" (to hit 2) is equivalent to starting a random walk initially at $R_0 = 0$, choose $a = 7$, $b = 1$, and use the formula for the probability of hitting a before hitting $-b$:

   $$p(a) = \frac{1 - (q/p)^b}{1 - (q/p)^{a+b}} = \frac{1 - (3/7)^1}{1 - (3/7)^8} = 0.572.$$

   (c) What is the probability that the stock will (as time goes on) become infinitely valuable without ever hitting 0?
   SOLUTION: Gambler's Ruin Problem, where we have $i = 3$ and want $P_i(\infty) = 1 - (q/p)^i = 1 - (3/7)^3 = 0.921$. (Because we have positive drift, this probability is positive.)

   (d) (*Continuation:*) Answer (a)– (c) in the case when the two probabilities 0.7 and 0.3 are reversed.
   SOLUTION:
   $q/p = 7/3$ now, and we have negative drift now.
   (a) 0.001
   (b) 0.002 (0.00152)
   (c) Since we now have negative drift, $P_i(\infty) = 0$.

   (e) (*Continuation:*) Answer (a)– (c) in the case when $p = q = 0.5$.
   SOLUTION: Now, $P_i(N) = i/N$ for the Gambler's ruin problem, and for the random walk with $R_0 = 0$, we have $P(a) = b/(a + b)$. This yields
   (a) 3/11
   (b) 1/8
   (c) Even when $p = 1/2$, we have $P_i(\infty) = 0$.

2. An insurance risk business has a reserve of money (in units of millions of dollars). Initially, it has 3 units. Every day, it earns 1 unit (interest), but also (each day) there is a chance of a claim against the business, independent of past days, of size 2 units with probability 0.35 (with probability 0.65 no such claim comes in). (A claim removes the 2 units from the reserve.)

   (a) What is the probability that the risk business will get ruined (run out of money)?

SOLUTION: Recall that the reserve process is exactly a simple random walk with $p = 0.65$ starting from $i = 3$. $(q/p = (.35/.65 = 7/13)$ Thus we want $1 - P_i(\infty)$, where $P_i(\infty) = \lim_{N\to\infty} P_i(N) = 1 - (q/p)^i$. Thus we want $(q/p)^i = (7/13)^3 = 0.156$.

(b) What is the smallest value i (units) the business would need to have started with to ensure that the probability of ruin is *less* than 1/2?

SOLUTION: Noting that $(7/13)^1 = 0.538 > 1/2$ and $(7/13)^2 = 0.29 < 1/2$, we see that the answer is $i = 2$.

3. A jumping bean moves on the integers according to a simple random walk taking one step per unit time. $R_n =$ the position at time $n \geq 0$. Assume that $p = 0.35$; the probability that a step takes the bean forward (to the right), and $q = 1 - p = 0.65$ is the probability that a step takes the bean backward (to the left). It starts off initially at position $R_0 = 5$.

(a) Does this random walk have positive drift or negative drift?

SOLUTION:

Negative drift (by definition) since $p < 1/2$; $P(\lim_{n\to\infty} R_n = -\infty) = 1$.

(b) What is the probability that the bean will go down to 0 before ever reaching \$6?

SOLUTION: With $q/p = 13/7$; $1 - P_5(6) = 0.473$, where

$$P_5(6) = \frac{1 - (q/p)^5}{1 - (q/p)^6} = 0.527.$$

(c) What is the probability that the bean will go below ($<$) 0 before ever reaching \$6?

SOLUTION:

Going below 0 means that it hits -1 since this is a simple random walk, only taking ± 1 size steps. So we want the probability that such a random walk, goes down by $b = 6$ before going up by $a = 1$.

So we want $1 - p(a)$, where (with $a = 1$, $b = 6$),

$$p(a) = \frac{1 - (\frac{q}{p})^b}{1 - (\frac{q}{p})^{a+b}}. \tag{1}$$

Calculation gives answer 0.468.

(d) What is the probability that the bean will *never* reach as high as 6.00?

SOLUTION: We want the probability that the random walk, starting at 5 never goes up by 1 to 6. This is equivalent to the random walk, starting initially at the origin, $R_0 = 0$, never reaching as high as 1, meaning that it never goes above the origin. Letting $M = \max_{n\geq 0} R_n$, (with $R_0 = 0$) we know that because the random walk has negative drift (e.g., $p < 1/2$),M has a geometric distribution, $P(M \geq a) = (p/q)^a$, $a \geq 0$. $(p/q = 7/13.)$ Thus we want $P(M \leq 0) = P(M = 0) = 1 - (p/q) = 4/13$.

4. As a more realistic model for asset pricing, suppose that the price of an asset moves (day by day) as

$$X_n = 8 \times 2^{R_n},\ n \geq 0,$$

where $R_0 = 0$, and $R_n = \sum_{k=1}^{n} \Delta_k,\ k \geq 1$, is a simple symmetric random walk; $P(\Delta = 1) = 1/2 = P(\Delta = -1)$.

   (a) What is the probability that the asset price reaches a high of 32 before a low of 1/2?
   SOLUTION: By taking logarithms in base 2, this is equivalent to the probability that $3+R_n$ hits 5 before -1, or that R_n (starting at 0) hits $a = 2$ before $-b = -4$: (Recall that $p = 1/2$)

$$p(a) = \frac{b}{a+b} = 2/3. \tag{2}$$

   (b) What is the probability that the asset price will ever reach as high as 2^{500}?
   SOLUTION: We want the probability that $3 + R_n$ ever hits 500, that is, the probability that R_n ever hits 497. Recall that for the simple symmetric random walk R_n, $P(M = \infty) = 1$ (and $P(m = -\infty) = 1$ meaning that it will hit any integer, however large (or however small), with certainty if we wait long enough. Thus the answer is 1.

5. Let $\{Y_n : n \geq 0\}$ be an i.i.d. sequence of r.v.s. and let $a_j \stackrel{\text{def}}{=} P(Y = j),\ -\infty < j < \infty$. Define

$$m_n \stackrel{\text{def}}{=} \min\{Y_0, \ldots, Y_n\},\ n \geq 0.$$

Show that $\{m_n\}$ forms a Markov chain by expressing it as a recursion.
SOLUTION: $m_{n+1} = \min\{m_n, Y_{n+1}\},\ n \geq 0.$

   (a) *The transition probablitities for* m_n are computed as $P(\min(i, Y) = j)$ while considering the 3 cases:
   $P_{i,j} = P(Y = j) = a_j$ for $j < i$;
   $P_{i,i} = P(Y \geq i) = \sum_{k \geq i} a_k$;
   $P_{i,j} = 0$ for $i < j$ (because m_n can never increase)

   (b) With probability 1 (wp1), $m_n \to -\infty$ as $n \to \infty$ because of the assumption that $a_j > 0$ for all $j < 0$: for every $j < 0$, no matter how small $a_j > 0$ is, there will always appear (wp1, for n large enough) a Y_n for which $Y_n = j$. This is a direct consequence of the strong law of large numbers: $wp1$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} I\{Y_n = j\} = P(Y = j) = a_j > 0.$$

   This implies that $Y_n = j$ for infinitely many values of n (otherwise the limit would be 0).

(c) In this case $a_j = 0, j < -3$ so $m_n \to -3$ as $n \to \infty$. $P_{i,j} = P(Y = j) = a_j = 1/7$ for $-3 < j < i \leq 3$;
$P_{i,i} = P(Y \geq i) = \sum_{k=i}^{3} a_k = (3 - i + 1)/7$ for $-3 \leq i \leq 3$;
$P_{i,j} = 0$ for $i < j$ (because m_n can never increase)

In matrix form,

$$P = \begin{array}{c} \\ (-3) \\ (-2) \\ (-1) \\ (0) \\ (1) \\ (2) \\ (3) \end{array} \begin{array}{c} \begin{array}{ccccccc} (-3) & (-2) & (-1) & (0) & (1) & (2) & (3) \end{array} \\ \left\| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/7 & 6/7 & 0 & 0 & 0 & 0 & 0 \\ 1/7 & 1/7 & 5/7 & 0 & 0 & 0 & 0 \\ 1/7 & 1/7 & 1/7 & 4/7 & 0 & 0 & 0 \\ 1/7 & 1/7 & 1/7 & 1/7 & 3/7 & 0 & 0 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 2/7 & 0 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \end{array} \right\| \end{array}$$

6. Let $X_n \stackrel{\text{def}}{=} Y_{n-1} + Y_n,\ n \geq 1,\ X_0 \stackrel{\text{def}}{=} 0$, where $\{Y_n : n \geq 0\}$ is an iid sequence of rvs with a 0.5 Bernoulli distribution: $P(Y = 0) = P(Y = 1) = 0.5$. Is $\{X_n\}$ a Markov chain? Either prove it is or show why it is not.

SOLUTION: Not a MC. For suppose that $X_n = Y_{n-1}+Y_n = 1$. Then either $Y_n = 1$ and $Y_{n-1} = 0$, or $Y_n = 0$ and $Y_{n-1} = 1$. To predict the value of $X_{n+1} = Y_{n+1} + Y_n$, we would need to know which of the two cases ocurred; this we could only determine by also knowing past values such as those of X_{n-1}. For example, if $X_{n-1} = 0$, then $Y_{n-1} = 0$, where as if $X_{n-1} = 2$, then $Y_{n-1} = 1$:

$P(X_{n+1} = 0 | X_n = 1, X_{n-1} = 0) = P(Y_{n+1} + Y_n = 0 | Y_n = 1) = P(Y_{n+1} = -1) = 0$
$P(X_{n+1} = 0 | X_n = 1, X_{n-1} = 2) = P(Y_{n+1} + Y_n = 0 | Y_n = 0) = P(Y_{n+1} = 0) = 0.5$
Hence, given X_n, X_{n+1} is not independent of X_{n-1}; the Markov property is not satisfied.