IEOR 4106, HMWK 1, Professor Sigman

1. An asset price starts off initially at price \$3.00 at the end of a day (day 0), and at the end of each consecutive day, independent of the past, the price goes up by one dollar (with probability $p = 0.7$) or down by one dollar (with probability $q = 0.3$).

   (a) What is the probability that the stock will reach \$11.00 before going down to 0?

   (b) What is the probability that the stock will reach \$10.00 before going down to a low of \$2.00?

   (c) What is the probability that the stock will (as time goes on) become infinitely valuable without ever hitting 0?

   (d) (*Continuation:*) Answer (a)– (c) in the case when the two probabilities 0.7 and 0.3 are reversed.

   (e) (*Continuation:*) Answer (a)– (c) in the case when $p = q = 0.5$.

2. An insurance risk business has a reserve of money (in units of millions of dollars). Initially, it has 3 units. Every day, it earns 1 unit (interest), but also (each day) there is a chance of a claim against the business, independent of past days, of size 2 units with probability 0.35 (with probability 0.65 no such claim comes in). (A claim removes the 2 units from the reserve.)

   (a) What is the probability that the risk business will get ruined (run out of money)?

   (b) What is the smallest value i (units) the business would need to have started with to ensure that the probability of ruin is *less* than 1/2?

3. A jumping bean moves on the integers according to a simple random walk taking one step per unit time. $R_n =$ the position at time $n \geq 0$. Assume that $p = 0.35$; the probability that a step takes the bean forward (to the right), and $q = 1 - p = 0.65$ is the probability that a step takes the bean backward (to the left). It starts off initially at position $R_0 = 5$.

   (a) Does this random walk have positive drift or negative drift?

   (b) What is the probability that the bean will go down to 0 before ever reaching \$6?

   (c) What is the probability that the bean will go below (<) 0 before ever reaching \$6?

   (d) What is the probability that the bean will *never* reach as high as 6.00?

4. As a more realistic model for asset pricing, suppose that the price of an asset moves (day by day) as

$$X_n = 8 \times 2^{R_n}, \; n \geq 0,$$

   where $R_0 = 0$, and $R_n = \sum_{k=1}^{n} \Delta_k, \; k \geq 1$, is a simple symmetric random walk; $P(\Delta = 1) = 1/2 = P(\Delta = -1)$.

   (a) What is the probability that the asset price reaches a high of 32 before a low of 1/2?

(b) What is the probability that the asset price will ever reach as high as 2^{500}?

5. Let $\{Y_n : n \geq 0\}$ be an i.i.d. sequence of r.v.s. and let $a_j \stackrel{\text{def}}{=} P(Y = j), \ -\infty < j < \infty$. Define
$$m_n \stackrel{\text{def}}{=} \min\{Y_0, \ldots, Y_n\}, \ n \geq 0.$$
Show that $\{m_n\}$ forms a Markov chain by expressing it as a recursion.

   (a) Determine the transition probabilities $P_{i,j}$ in terms of the a_j.

   (b) Assume that $a_j > 0, \ -\infty < j < \infty$. Compute $\lim_{n\to\infty} m_n$.

   (c) Suppose instead that $a_j = P(Y = j) = 1/7, \ j \in \{-3, -2, -1, 0, 1, 2, 3\}$; $a_j = 0$ otherwise. (Thus Y has a discrete uniform distribution over the 7 point set.) Determine the transition probabilities $P_{i,j}$ and find $\lim_{n\to\infty} m_n$.

6. Let $X_n \stackrel{\text{def}}{=} Y_{n-1} + Y_n, \ n \geq 1, \ X_0 \stackrel{\text{def}}{=} 0$, where $\{Y_n : n \geq 0\}$ is an iid sequence of rvs with a 0.5 Bernoulli distribution: $P(Y = 0) = P(Y = 1) = 0.5$. Is $\{X_n\}$ a Markov chain? Either prove it is or show why it is not.