IEOR 4106, HMWK 2, Professor Sigman

1. Consider the *Rat in the Open Maze*; 4 rooms, and the outside (state 0), but now the probabilities are $P_{1,2} = 3/4,\ P_{1,3} = 1/4$, $P_{2,1} = 7/8,\ P_{2,4} = 1/8$; all the other probabilities are "equally likely" as before. Solve for $E(T_{3,0})$, the expected number of moves until the rat escapes given it starts in Room 3.

   Solution:

   Let $T_i = T_{i,0} = \min\{n \geq 1 : X_n = 0 \mid X_0 = i\},\ i = 1, 2, 3, 4$. Let $x_i = E(T_{i,0}) = E(T_i)$. We want x_3.

   Conditioning on the first step $X_1 = j$ (given $X_0 = 1, 2, 3, 4$ respectively) yields four linear equations with four unknowns

$$\begin{aligned}
E(T_1) &= \frac{3}{4}E(1+T_2) + \frac{1}{4}E(1+T_3) = 1 + \frac{3}{4}E(T_2) + \frac{1}{4}E(T_3) \\
E(T_2) &= \frac{7}{8}E(1+T_1) + \frac{1}{8}E(1+T_4) = 1 + \frac{7}{8}E(T_1) + \frac{1}{8}E(T_4) \\
E(T_3) &= \frac{1}{2}E(1+T_1) + \frac{1}{2}E(1+T_4) = 1 + \frac{1}{2}E(T_1) + \frac{1}{2}E(T_4) \\
E(T_4) &= \frac{1}{3}(1) + \frac{1}{3}E(1+T_3) + \frac{1}{3}E(1+T_2) = 1 + \frac{1}{3}E(T_2) + \frac{1}{3}E(T_3).
\end{aligned}$$

   In terms of x_i:

$$\begin{aligned}
x_1 &= 1 + \frac{3}{4}x_2 + \frac{1}{4}x_3 \\
x_2 &= 1 + \frac{7}{8}x_1 + \frac{1}{8}x_4 \\
x_3 &= 1 + \frac{1}{2}x_1 + \frac{1}{2}x_4 \\
x_4 &= 1 + \frac{1}{3}x_2 + \frac{1}{3}x_3
\end{aligned}$$

   Solving yields

$$\begin{aligned}
x_1 &= \frac{187}{7} \\
x_2 &= \frac{186}{7} \\
x_3 &= \frac{162}{7} \\
x_4 &= \frac{123}{7}
\end{aligned}$$

2. *Continuation:* Consider the same maze as in (1), but now there is no escape; that is, no state 0. Instead, the rat wanders around the 4 rooms forever. Use the same probabilities as in (1) except from room 4: $P_{4,3} = P_{4,2} = 1/2$. Let

$T_{1,4} = \min\{n \geq 1 : X_n = 4 \mid X_0 = 1\}$, denote the number of moves until reaching Room 4 (for the first time) given the rats starts off in Room 1. Compute $E(T_{1,4})$.

Solution:

Let $T_i = \min\{n \geq 1 : X_n = 4 \mid X_0 = i\}$, $i = 1, 2, 3$. Let $x_i = E(T_i)$. We want $x_1 = E(T_1)$.

Conditioning on the first step $X_1 = j$ (given $X_0 = 1, 2, 3$ respectively) yields three linear equations with three unknowns

$$\begin{aligned} E(T_1) &= \frac{3}{4}E(1+T_2) + \frac{1}{4}E(1+T_3) = 1 + \frac{3}{4}E(T_2) + \frac{1}{4}E(T_3) \\ E(T_2) &= \frac{7}{8}E(1+T_1) + \frac{1}{8}(1) = 1 + \frac{7}{8}E(T_1) \\ E(T_3) &= \frac{1}{2}E(1+T_1) + \frac{1}{2}(1) = 1 + \frac{1}{2}E(T_1) \end{aligned}$$

In terms of x_i:

$$\begin{aligned} x_1 &= 1 + \frac{3}{4}x_2 + \frac{1}{4}x_3 \\ x_2 &= 1 + \frac{7}{8}x_1 \\ x_3 &= 1 + \frac{1}{2}x_1 \end{aligned}$$

Solving yields

$$\begin{aligned} x_1 &= \frac{64}{7} \\ x_2 &= 9 \\ x_3 &= \frac{39}{7} \end{aligned}$$

3. Consider the Gambler's ruin problem Markov chain with $\mathcal{S} = \{0, 1, 2, \ldots, N\}$, but for which the value of p depends upon i, $1 \leq i \leq N-1$. That is, if $X_n = i$, then (independent of the past) the probability that the Gambler wins \$1 is p_i, the probability the Gambler loses \$1 is $q_i = 1 - p_i$. (As before $P_{0,0} = P_{N,N} = 1$.) Each p_i satisfies $0 < p_i < 1$. Let $P_i(N)$ denote the probability that the Gambler, starting with $X_0 = i$, will reach N before 0.

   (a) For $N = 3$, explicitly solve for the $P_i = P_i(N), 1 \leq i \leq 2$. (Recall the boundary conditions $P_0 = 0$, $P_N = 1$.)

   SOLUTION:

   Given $X_0 = 1$, we condition on the first gamble to obtain

   $$P_1 = p_1 P_2 + q_1 P_0 = p_1 P_2.$$

   Similarly for $X_0 = 2$,

   $$P_2 = p_1 P_3 + q_2 P_1 = p_2 + q_2 P_1.$$

Thus we have 2 equations with 2 unknowns:

$$
\begin{aligned}
P_1 &= p_1 P_2 && (1)\\
P_2 &= p_2 + q_2 P_1. && (2)
\end{aligned}
$$

Solving yields:

$$
\begin{aligned}
P_2 &= \frac{p_2}{1 - q_2 p_1} && (3)\\
P_1 &= \frac{p_1 p_2}{1 - q_2 p_1}. && (4)
\end{aligned}
$$

(b) Show that if $p_1 = p_2 = 1/2$ in (a), then you get the same answer (e.g., plug in and see) as for the regular Gambler's ruin problem.

SOLUTION:

It is immediate that plugging in $p_1 = p_2 = 1/2$ into the solution in (a) that we get $P_1 = 1/3,\ P_2 = 2/3$ as should be.

4. Consider from class lecture, the weather Markov chain $X_n = (W_{n-1}, W_n)$ where $W_n \in \{0, 1\}$ (1 = rain, 0 = no rain), and the labeling is given by

$$
\begin{aligned}
0 &= (0,0)\\
1 &= (0,1)\\
2 &= (1,0)\\
3 &= (1,1);
\end{aligned}
$$

the state space is thus $\mathcal{S} = \{0, 1, 2, 3\}$. We had given probabilities leading to the 1-step transition matrix

$$
P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}.
$$

Compute the probability that it does not rain 2 days from now, given that it rained today but not yesterday. (Think of now as Monday, yesterday as Sunday, and 2 days from now as Wednesday if you so wish.)

SOLUTION:

We want

$$P(X_2 = (0,0) = 0 \text{ or } (1,0) = 2 \mid X_0 = (0,1) = 1).$$

Equivalently, we want

$$P^{(2)}_{1,0} + P^{(2)}_{1,2}.$$

Computing $P^{(2)} = P^2$ yields:

$$P^2 = \begin{pmatrix} .64 & .16 & .1 & .1 \\ .3 & .2 & .15 & .35 \\ .48 & .12 & .20 & .20 \\ .18 & .12 & .21 & .49 \end{pmatrix}.$$

$P^{(2)}_{1,0} + P^{(2)}_{1,2} = .3 + .15 = .45$

5. Consider modeling the weather where we now assume that the weather today depends (at most) on the previous *three* days weather instead of 2 as we did in Problem 3 above. Letting W_n denote weather on the n^{th} day (0 = no rain, 1 = rain), let $X_n = (W_{n-2}, W_{n-1}, W_n)$. There are 8 states, and we will relabel them 0—7 as: $(0,0,0) = 0,\ (1,0,0) = 1,\ (0,1,0) = 2,\ (0,0,1) = 3,\ (1,1,0) = 4,\ (1,0,1) = 5,\ (0,1,1) = 6,\ (1,1,1) = 7$. We will *assume* it forms a Markov chain. Assume that if it has rained for the past 3 days, then it will rain today with probability 0.8; if it did not rain on any of the past three days, then it will rain today with probability 0.10. In any other case assume that the weather today will with probability 0.7 be the same as the weather yesterday. Derive the transition matrix.

   SOLUTION: A state is in order (left to right) of "two days ago" "yesterday", "today". For instance, (110) means that it rained the day before yesterday, and it rained yesterday but it did not rain today. Moving ahead in time by one day (n to $n+1$) the state (110) becomes of the form $(1, 0, x)$, where $x = 0, 1$; so it becomes either $(1, 0, 0)$ or $(1, 0, 1)$. So there are only two possibilities for each transition. This means that each row of the transition matrix will have only two non-zero elements.
   We re-label the states $0 - 7$. The info given, for example, "if it has rained for the past 3 days, then it will rain today with probability 0.8" means that $P(X_{n+1} = (1,1,1) \mid X_n = (1,1,1)) = 0.8$ and $P(X_{n+1} = (1,1,0) \mid X_n = (1,1,1)) = 0.2$. In our re-labeling this becomes $P(X_{n+1} = 7 \mid X_n = 7) = 0.8$ and $P(X_{n+1} = 4 \mid X_n = 7) = 0.2$, yielding the last row of the matrix below. Similarly, "if it did not rain on any of the past three days, then it will rain today with probability 0.10" yields $P(X_{n+1} = (0,0,1) \mid X_n = (0,0,0)) = 0.1$ and $P(X_{n+1} = (0,0,0) \mid X_n = (0,0,0)) = 0.9$, or $P(X_{n+1} = 3 \mid X_n = 0) = 0.1$ and $P(X_{n+1} = 0 \mid X_n = 0) = 0.9$; yielding the initial row of the matrix. The others rows are derived in a similar way, each with a .7 and a .3 element.

$P =$

	(000)	(100)	(010)	(001)	(110)	(101)	(011)	(111)
(000)	.9	0	0	.1	0	0	0	0
(100)	.7	0	0	.3	0	0	0	0
(010)	0	.7	0	0	0	.3	0	0
(001)	0	0	.3	0	0	0	.7	0
(110)	0	.7	0	0	0	.3	0	0
(101)	0	0	.3	0	0	0	.7	0
(011)	0	0	0	0	.3	0	0	.7
(111)	0	0	0	0	.2	0	0	.8

6. *Continuation:*

Given that it rained today, rained yesterday and rained the day before yesterday, compute the probability that it does not rain 2 days from now.

SOLUTION: We want

$$P(X_2 = (1,0,0) = 1 \text{ or } (1,1,0) = 4 \mid X_0 = (1,1,1) = 7).$$

Equivalently, we want

$$P^{(2)}_{7,1} + P^{(2)}_{7,4}.$$

Recall that

$$P = \begin{array}{c} \\ (000) \\ (100) \\ (010) \\ (001) \\ (110) \\ (101) \\ (011) \\ (111) \end{array} \begin{array}{cccccccc} (000) & (100) & (010) & (001) & (110) & (101) & (011) & (111) \\ .9 & 0 & 0 & .1 & 0 & 0 & 0 & 0 \\ .7 & 0 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & .7 & 0 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & .3 & 0 & 0 & 0 & .7 & 0 \\ 0 & .7 & 0 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & .3 & 0 & 0 & 0 & .7 & 0 \\ 0 & 0 & 0 & 0 & .3 & 0 & 0 & .7 \\ 0 & 0 & 0 & 0 & .2 & 0 & 0 & .8 \end{array}$$

Computing $P^{(2)} = P^2$ yields:

$$P^2 = \begin{array}{c} \\ (000) \\ (100) \\ (010) \\ (001) \\ (110) \\ (101) \\ (011) \\ (111) \end{array} \begin{array}{cccccccc} (000) & (100) & (010) & (001) & (110) & (101) & (011) & (111) \\ .81 & 0 & .03 & .09 & 0 & 0 & .07 & 0 \\ .63 & 0 & .09 & .07 & 0 & 0 & .21 & 0 \\ .49 & 0 & .09 & .21 & 0 & 0 & .21 & 0 \\ 0 & .21 & 0 & 0 & .21 & .09 & 0 & .49 \\ 0 & .21 & 0 & 0 & .14 & .09 & 0 & .56 \\ 0 & .14 & 0 & 0 & .16 & .06 & 0 & .64 \\ 0 & 0 & 0 & 0 & .3 & 0 & 0 & .7 \\ 0 & .14 & 0 & 0 & .16 & .06 & 0 & .64 \end{array}$$

$P^{(2)}_{7,1} + P^{(2)}_{7,4} = .14 + .16 = 0.3$

7. For the Gambler's ruin problem, with $N = 3$ and $p = 0.3$: Suppose $X_0 = 1$. Compute the probability that the Gambler stops gambling by ($\leq$) time 5. (Recall the Markov chain for this model, in which $P_{0,0} = P_{N,N} = 1$.)

   SOLUTION: We must first compute $P^5 = (P^{(5)}_{i,j})$ for the 4×4 transition matrix P, and then we want $P^{(5)}_{1,0} + P^{(5)}_{1,3}$.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$P^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.8779 & 0 & 0.0132 & 0.1089 \\ 0.5929 & 0.0309 & 0 & 0.3762 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$P^{(5)}_{1,0} + P^{(5)}_{1,3} = 0.8779 + 0.1089 = 0.9868.$

8. Consider the Binomial Lattice Model (BLM), $S_n = S_0 Y_1 \cdots Y_n$, where $S_0 = 50$.

   Suppose that $p = 0.5$, and $u = 1.8$ and $d = 0.5$. Show that $E(S_n) \to \infty$ as $n \to \infty$, but in fact $S_n \to 0$ as $n \to \infty$ with probability 1.

   In other words: You will become infinitely rich on average, but with certainty will go broke!! (Interesting , yes?) HINT: To show that $S_n \to 0$, take natural logarithms of S_n first.........

   SOLUTION: $E(S_n) = 50E(Y)^n = 50(1.15)^n \to \infty$, $(E(Y) = pu + (1-p)d = 1.15 > 1)$. Let

$$R_n = \ln(S_n) = \ln(50) + \sum_{i=1}^{n} \ln(Y_i).$$

   A random walk with iid increments distributed as $\Delta = \ln(Y)$. It suffices to show negative drift, $E(\Delta) < 0$, for then $R_n \to -\infty$ wp1, and hence $S_n = e^{R_n} \to 0$. To this end: $E(\ln(Y)) = (0.5)\ln(1.8) + (0.5)\ln(0.5) = (0.5)\ln(0.9) < 0$.