IEOR 4106, HMWK 2, Professor Sigman

1. Consider the *Rat in the Open Maze*; 4 rooms, and the outside (state 0), but now the probabilities are $P_{1,2} = 3/4$, $P_{1,3} = 1/4$, $P_{2,1} = 7/8$, $P_{2,4} = 1/8$; all the other probabilities are "equally likely" as before. Solve for $E(T_{3,0})$, the expected number of moves until the rat escapes given it starts in Room 3.

2. *Continuation:* Consider the same maze as in (1), but now there is no escape; that is, no state 0. Instead, the rat wanders around the 4 rooms forever. Use the same probabilities as in (1) except from room 4: $P_{4,3} = P_{4,2} = 1/2$. Let $T_{1,4} = \min\{n \geq 1 : X_n = 4 \mid X_0 = 1\}$, denote the number of moves until reaching Room 4 (for the first time) given the rats starts off in Room 1. Compute $E(T_{1,4})$.

3. Consider the Gambler's ruin problem Markov chain with $\mathcal{S} = \{0, 1, 2, \ldots, N\}$, but for which the value of p depends upon i, $1 \leq i \leq N - 1$. That is, if $X_n = i$, then (independent of the past) the probability that the Gambler wins \$1 is p_i, the probability the Gambler loses \$1 is $q_i = 1 - p_i$. (As before $P_{0,0} = P_{N,N} = 1$.) Each p_i satisfies $0 < p_i < 1$. Let $P_i(N)$ denote the probability that the Gambler, starting with $X_0 = i$, will reach N before 0.

   (a) For $N = 3$, explicitly solve for the $P_i(N), 1 \leq i \leq 2$.

   (b) Show that if $p_1 = p_2 = 1/2$ in (a), then you get the same answer (e.g., plug in and see) as for the regular Gambler's ruin problem.

4. Consider from class lecture, the weather Markov chain $X_n = (W_{n-1}, W_n)$ where $W_n \in \{0, 1\}$ (1 = rain, 0 = no rain), and the labeling is given by

$$\begin{aligned} 0 &= (0,0) \\ 1 &= (0,1) \\ 2 &= (1,0) \\ 3 &= (1,1); \end{aligned}$$

   the state space is thus $\mathcal{S} = \{0, 1, 2, 3\}$. We had given probabilities leading to the 1-step transition matrix

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}.$$

   Compute the probability that it does not rain 2 days from now, given that it rained today but not yesterday. (Think of now as Monday, yesterday as Sunday, and 2 days from now as Wednesday if you so wish.)

5. Consider modeling the weather where we now assume that the weather today depends (at most) on the previous *three* days weather instead of 2 as we did in Problem 3 above. Letting W_n denote weather on the n^{th} day (0 = no rain, 1 = rain), let $X_n = (W_{n-2}, W_{n-1}, W_n)$. There are 8 states, and we will relabel them 0—7 as: $(0, 0, 0) = 0$, $(1, 0, 0) = 1$, $(0, 1, 0) = 2$, $(0, 0, 1) = 3$, $(1, 1, 0) = 4$, $(1, 0, 1) =$

5, $(0,1,1) = 6$, $(1,1,1) = 7$. We will *assume* it forms a Markov chain. Assume that if it has rained for the past 3 days, then it will rain today with probability 0.8; if it did not rain on any of the past three days, then it will rain today with probability 0.10. In any other case assume that the weather today will with probability 0.7 be the same as the weather yesterday. Derive the transition matrix.

6. *Continuation:*

   Given that it rained today, rained yesterday and rained the day before yesterday, compute the probability that it does not rain 2 days from now.

7. For the Gambler's ruin problem, with $N = 3$ and $p = 0.3$: Suppose $X_0 = 1$. Compute the probability that the Gambler stops gambling by ($\leq$) time 5. (Recall the Markov chain for this model, in which $P_{0,0} = P_{N,N} = 1$.)

8. Consider the Binomial Lattice Model (BLM), $S_n = S_0 Y_1 \cdots Y_n$, where $S_0 = 50$.

   Suppose that $p = 0.5$, and $u = 1.8$ and $d = 0.5$. Show that $E(S_n) \to \infty$ as $n \to \infty$, but in fact $S_n \to 0$ as $n \to \infty$ with probability 1.

   In other words: You will become infinitely rich on average, but with certainty will go broke!! (Interesting , yes?) HINT: To show that $S_n \to 0$, take natural logarithms of S_n first.........