IEOR 4106, HMWK 3, Professor Sigman

1. Each of the following transition matrices is for a Markov chain. For each, find the communication classes for breaking down the state space, $\mathcal{S} = C_1 \cup C_2 \cup \cdots$ and for each class C_k tell if it is recurrent or transient.

   (a)
$$P = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/2 \\ 7/8 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/7 & 0 & 6/7 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (b)
$$P = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/10 & 3/10 & 2/10 & 4/10 \\ 0 & 6/11 & 0 & 5/11 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (c)
$$P = \begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (d)
$$P = \begin{pmatrix} 1/7 & 0 & 2/7 & 0 & 4/7 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/5 & 0 & 1/5 & 0 & 3/5 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/4 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3, 4\}$.

   (e)
$$P = \begin{pmatrix} 1/9 & 2/9 & 1/9 & 1/9 & 4/9 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 6/7 & 0 & 0 & 1/7 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 2/11 & 0 & 5/11 & 0 & 4/11 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3, 4\}$.

   SOLUTION:

   (a) This MC is irreducible, all states communicate; $C = \{0, 1, 2, 3\} (recurrent)$

(b) There are 3 communication classes, $C_1 = \{0\}(transient), C_2 = \{1,3\}(recurrent), C_3 = \{2\}(transient)$

(c) There are 3 communication classes, $C_1 = \{0\}(transient), C_2 = \{1\}(transient), C_3 = \{2,3\}(recurrent)$

(d) There are 2 communication classes, $C_1 = \{0,2,4\}(recurrent), C_2 = \{1,3\}(recurrent)$

(e) This MC is irreducible, $C = \{0,1,2,3,4\}(recurrent)$

2. Consider a Markov chain on $\mathcal{S} = \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ with transitions as follows: For $0 < p < 1$ fixed, $P_{i,i+3} = p,\ P_{i,i-3} = 1-p,\ i \in \mathbb{Z}$. This is a random walk with jumps of size ± 3 (instead of ± 1). Give the communications classes and tell if they are recurrent or transient when $p \neq 1/2$, and for $p = 1/2$.

SOLUTION:

Observe that this is a random walk $\{R_n\}$ in which the jumps are of size ± 3 instead of ± 1. This means that if $R_0 = 0$ then the only states that can be visited are $C_1 = \{\pm 3k : k \geq 0\}$, if $R_0 = 1$, then the only states that can be visited are $C_2 = \{1 \pm 3k : k \geq 0\}$, and finally if $R_0 = 2$, then the only states that can be visited are $C_3 = \{2 \pm 3k : k \geq 0\}$. Each of the three C_i is a communication class and $\mathcal{S} = \mathbb{Z} = C_1 \cup C_2 \cup C_3$.

Similar to the ± 1 case, $R_n \to +\infty$ when $p > 1/2$, $R_n \to -\infty$ when $p < 1/2$ (via the Strong Law of Large Numbers). Thus in these cases, each C_i is transient. When $p = 1/2$, each C_i is recurrent.

3. Consider a Markov chain $\{X_n : n \geq 0\}$ with $\mathcal{S} = \{0,1,2\}$, and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/5 & 4/5 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

(a) Suppose that (independently) X_0 is chosen randomly with $P(X_0 = 0) = P(X_0 = 1) = 1/8,\ P(X_0 = 2) = 3/4$. Compute $E(X_3)$.

SOLUTION: We will use the law of total probability:

$$E(X_3) = \sum_{i=0}^{2} E(X_3 \mid X_0 = i) P(X_0 = i).$$

Noting that $P(X_3 = j \mid X_0 = i) = P_{i,j}^{(3)}$, we have by definition,

$$E(X_3 \mid X_0 = i) = \sum_{j=0}^{2} j P_{i,j}^{(3)},$$

the mean of the i^{th} row of $P^{(3)} = P^3$.
Thus we first must compute

$$P^3 = \begin{pmatrix} 331/1080 & 401/2700 & 109/200 \\ 82/225 & 109/1125 & 202/375 \\ 13/36 & 41/270 & 263/540 \end{pmatrix}.$$

Then we get:

$$m_0 = E(X_3 \mid X_0 = 0) = \sum_{j=0}^{2} j P_{0,j}^{(3)} = 836/675, \quad (1)$$

$$m_1 = E(X_3 \mid X_0 = 1) = \sum_{j=0}^{2} j P_{1,j}^{(3)} = 1321/1125, \quad (2)$$

$$m_2 = E(X_3 \mid X_0 = 2) = \sum_{j=0}^{2} j P_{2,j}^{(3)} = 152/135. \quad (3)$$

Finally we get

$$E(X_3) = \sum_{i=0}^{3} m_i P(X_0 = i) = 30943/27000 = 1.146.$$

(b) Show that the chain is irreducible and solve for the limiting distribution: Solve $\pi = \pi P$ for the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2)$, where $\pi_j > 0,\ j \in \mathcal{S}$, and $\sum_{j \in \mathcal{S}} \pi_j = 1$.
SOLUTION: Irreducibility follows since $0 \to 1 \to 2 \to 0$ with probability $P_{0,1}P_{1,2}P_{2,0} = (1/3)(4/5)(1/3) > 0$.
The $\pi = \pi P$ equations:

$$\begin{aligned}
\pi_0 &= (1/2)\pi_0 + (1/3)\pi_2 \\
\pi_1 &= (1/3)\pi_0 + (1/5)\pi_1 \\
\pi_2 &= (1/6)\pi_0 + (4/5)\pi_1 + (2/3)\pi_2
\end{aligned}$$

(And we can also use $\pi_0 + \pi_1 + \pi_2 = 1$.)
Solving we get

$$\begin{aligned}
\pi_0 &= 12/35, \\
\pi_1 &= 1/7, \\
\pi_2 &= 18/35.
\end{aligned}$$

(c) Compute the variance, given by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n^2 \; - \; \left[\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n \right]^2$$

SOLUTION: Time average = mean of stationary distribution, the first 2 moments are:

$$m_1 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n = \sum_{j=0}^{2} j\pi_j = 41/35.$$

$$m_2 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} X_n^2 = \sum_{j=0}^{2} j^2\pi_j = 11/5.$$

The answer is thus $m_2 - m_1^2 = 1014/1225 = 0.8278$.

(d) Given that the chain is now in state 2, what is the expected amount of time until the chain returns to state 2?
SOLUTION: $E(\tau_{2,2}) = \frac{1}{\pi_2} = 35/18$.

4. Consider a simple random walk with $0 < p < 1$. But now we restrict it to be non-negative in the following way: $P_{0,1} = 1$, and $P_{i,0} = q = 1-p,\ i \geq 1$; otherwise $P_{i,i+1} = p,\ i \geq 1$ and $P_{i,i-1} = 0,\ i \geq 2$. Thus the state space is still infinite but non-negative; $\mathcal{S} = \{0, 1, 2, \ldots\}$. Imagine that this Markov chain X_n represents the total fortune of a gambler after his n^{th} gamble, where whenever he goes broke, he is given a dollar by a friend so that he can keep gambling. Also note that with probability q, he might go broke after only one gamble no matter what his total fortune is.

(a) Show that this chain is irreducible.

SOLUTION: $P_{0,1} = 1$ and for any $j > i \geq 1$, $P_{i,j}^{(j-i)} = p^{j-i} > 0$. If $1 < j < i$, then the chain goes via state 0: $i \to 0 \to j$: $P_{i,j}^{(1+j)} = qp^{j-1} > 0$. Thus all states communicate.

(b) Show (via solving $\pi = \pi P$) that it is positive recurrent (for all $0 < p < 1$). HINT: Make sure to use the equation $\sum_{j=0}^{\infty} \pi_j = 1$.

SOLUTION: Note that the first equation is very simple and gives us immediately the value of π_0:

$$\pi_0 = 0\pi_0 + q\pi_1 + \cdots + = q(\sum_{j=1}^{\infty} \pi_j) = q(1-\pi_0),$$

since we must have $\sum_{j=0}^{\infty} \pi_j = 1$ yielding $\sum_{j=1}^{\infty} \pi_j = 1 - \pi_0$. Thus we get $\pi_0(1+q) = q$ yielding $\pi_0 = q/(1+q)$.
The next equation is simply $\pi_1 = \pi_0$, and all the others are $\pi_{n+1} = p\pi_n,\ n \geq 1$. Thus we get $\pi_n = p^{n-1}\pi_0 = p^{n-1}(q/(1+q)),\ n \geq 1$.

(c) What is the long-run proportion of time (gambles) that the gambler goes broke?
SOLUTION: As we already showed, $\pi_0 = q/(1+q)$.

(d) Suppose that $p = 0.70$. Given that the gambler has exactly \$5 now, on average how many gambles will it be until he has \$5 again?
SOLUTION:
$E(T_{5,5}) = 1/\pi_5$
$= 1/[p^4(q/(1+q))] = 1/[(.7)^4(.3/(1+.3))] = 18.048$

5. Consider the simple random walk $\{R_n\}$ with $0 < p < 1$ and in which $p \neq q$. We know that this Markov chain is irreducible and transient (all states are transient).

Thus for each state $i \in \mathbb{Z}$, $f_i < 1$ where f_i = the probability that the chain will ever return back to state i given that $R_0 = i$. The objective of this problem is to exactly compute f_i. Note that f_i is the same for all i. So it suffices to derive f_0. So we assume that $R_0 = 0$.

(a) Letting $M = \max_{n \geq 0} R_n$, and $m = \min_{n \geq 0} R_n$, argue that

$$f_0 = pP(m \leq -1) + qP(M \geq 1).$$

SOLUTION:
Condition on $R_1 = \Delta_1 = \pm 1$: If $R_1 = 1$ (probability p) then the only way the chain can revisit state 0 is if it goes down by at least 1; equivalently if $m \leq -1$. If $R_1 = -1$ (probability q), then the only way the chain can revisit state 0 is if it goes up by at least 1; equivalently if $M \geq 1$.

(b) From (a), solve for f_0 for the two cases $p < q$ and $p > q$.
SOLUTION: $p < q$. In this case we know that $P(M \geq a) = (p/q)^a,\ a \geq 0$, and $P(m \leq -b) = 1,\ b \geq 0$. Thus we get

$$f_0 = pP(m \leq -1) + qP(M \geq 1) = p \times 1 + q(p/q)^1 = 2p.$$

Similarly, if $q < p$, then $P(M \geq a) = 1,\ a \geq 0$ and $P(m \leq -b) = (q/p)^b,\ b \geq 0$. We get

$$f_0 = p(q/p)^1 + q \times 1 = 2q.$$