IEOR 4106, HMWK 3, Professor Sigman

1. Each of the following transition matrices is for a Markov chain. For each, find the communication classes for breaking down the state space, $S = C_1 \cup C_2 \cup \cdots$ and for each class $C_k$ tell if it is recurrent or transient.

   (a) $P = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/2 \\ 7/8 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/7 & 0 & 6/7 \end{pmatrix}$.

   $S = \{0, 1, 2, 3\}$.

   (b) $P = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/10 & 3/10 & 2/10 & 4/10 \\ 0 & 6/11 & 0 & 5/11 \end{pmatrix}$.

   $S = \{0, 1, 2, 3\}$.

   (c) $P = \begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$.

   $S = \{0, 1, 2, 3\}$.

   (d) $P = \begin{pmatrix} 1/7 & 0 & 2/7 & 0 & 4/7 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/5 & 0 & 1/5 & 0 & 3/5 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/4 \end{pmatrix}$.

   $S = \{0, 1, 2, 3, 4\}$.

   (e) $P = \begin{pmatrix} 1/9 & 2/9 & 1/9 & 1/9 & 4/9 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 6/7 & 0 & 0 & 1/7 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 2/11 & 0 & 5/11 & 0 & 4/11 \end{pmatrix}$.

   $S = \{0, 1, 2, 3, 4\}$.

2. Consider a Markov chain on $S = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with transitions as follows: For $0 < p < 1$ fixed, $P_{i,i+3} = p$, $P_{i,i-3} = 1 - p$, $i \in \mathbb{Z}$. This is a random walk with jumps of size $\pm 3$ (instead of $\pm 1$). Give the communications classes and tell if they are recurrent or transient when $p \neq 1/2$, and for $p = 1/2$. 

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3. Consider a Markov chain \( \{X_n : n \geq 0\} \) with \( \mathcal{S} = \{0, 1, 2\} \), and transition matrix 
\[
P = \begin{pmatrix}
1/2 & 1/3 & 1/6 \\
0 & 1/5 & 4/5 \\
1/3 & 0 & 2/3
\end{pmatrix}.
\]
(a) Suppose that (independently) \( X_0 \) is chosen randomly with 
\( P(X_0 = 0) = P(X_0 = 1) = 1/8 \), \( P(X_0 = 2) = 3/4 \). Compute \( E(X_3) \).
(b) Show that the chain is irreducible and solve for the limiting distribution: Solve 
\( \pi = \pi P \) for the limiting distribution \( \pi = (\pi_0, \pi_1, \pi_2) \), where \( \pi_j > 0, j \in \mathcal{S} \), and \( \sum_{j \in \mathcal{S}} \pi_j = 1 \).
(c) Compute the variance, given by 
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n^2 \quad \text{with} \quad \left( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n \right)^2
\]
(d) Given that the chain is now in state 2, what is the expected amount of time until the chain returns to state 2?

4. Consider a simple random walk with \( 0 < p < 1 \). But now we restrict it to be non-negative in the following way: \( P_{0,1} = 1 \), and \( P_{i,0} = q = 1 - p, \; i \geq 1 \); otherwise \( P_{i,i+1} = p, \; i \geq 1 \) and \( P_{i,i-1} = 0, \; i \geq 2 \). Thus the state space is still infinite but non-negative; \( \mathcal{S} = \{0, 1, 2, \ldots\} \). Imagine that this Markov chain \( X_n \) represents the total fortune of a gambler after his \( n^{th} \) gamble, where whenever he goes broke, he is given a dollar by a friend so that he can keep gambling. Also note that with probability \( q \), he might go broke after only one gamble no matter what his total fortune is.
(a) Show that this chain is irreducible.
(b) Show (via solving \( \pi = \pi P \)) that it is positive recurrent (for all \( 0 < p < 1 \)).
HINT: Make sure to use the equation \( \sum_{j=0}^{\infty} \pi_j = 1 \).
(c) What is the long-run proportion of time (gambles) that the gambler goes broke?
(d) Suppose that \( p = 0.70 \). Given that the gambler has exactly $5 now, on average how many gambles will it be until he has $5 again?

5. Consider the simple random walk \( \{R_n\} \) with \( 0 < p < 1 \) and in which \( p \neq q \). We know that this Markov chain is irreducible and transient (all states are transient). Thus for each state \( i \in \mathbb{Z}, f_i < 1 \) where \( f_i \) = the probability that the chain will ever return back to state \( i \) given that \( R_0 = i \). The objective of this problem is to exactly compute \( f_i \). Note that \( f_i \) is the same for all \( i \). So it suffices to derive \( f_0 \). So we assume that \( R_0 = 0 \).
(a) Letting \( M = \max_{n \geq 0} R_n \), and \( m = \min_{n \geq 0} R_n \), argue that 
\[
f_0 = pP(m \leq -1) + qP(M \geq 1).
\]
(b) From (a), solve for \( f_0 \) for the two cases \( p < q \) and \( p > q \).