IEOR 4106, HMWK 3, Professor Sigman

1. Each of the following transition matrices is for a Markov chain. For each, find the communication classes for breaking down the state space, $\mathcal{S} = C_1 \cup C_2 \cup \cdots$ and for each class C_k tell if it is recurrent or transient.

   (a)
   $$P = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/2 \\ 7/8 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/7 & 0 & 6/7 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (b)
   $$P = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/10 & 3/10 & 2/10 & 4/10 \\ 0 & 6/11 & 0 & 5/11 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (c)
   $$P = \begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3\}$.

   (d)
   $$P = \begin{pmatrix} 1/7 & 0 & 2/7 & 0 & 4/7 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/5 & 0 & 1/5 & 0 & 3/5 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/4 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3, 4\}$.

   (e)
   $$P = \begin{pmatrix} 1/9 & 2/9 & 1/9 & 1/9 & 4/9 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 6/7 & 0 & 0 & 1/7 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 2/11 & 0 & 5/11 & 0 & 4/11 \end{pmatrix}.$$
   $\mathcal{S} = \{0, 1, 2, 3, 4\}$.

2. Consider a Markov chain on $\mathcal{S} = \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ with transitions as follows: For $0 < p < 1$ fixed, $P_{i,i+3} = p,\ P_{i,i-3} = 1 - p,\ i \in \mathbb{Z}$. This is a random walk with jumps of size ± 3 (instead of ± 1). Give the communications classes and tell if they are recurrent or transient when $p \neq 1/2$, and for $p = 1/2$.

3. Consider a Markov chain $\{X_n : n \geq 0\}$ with $\mathcal{S} = \{0, 1, 2\}$, and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/5 & 4/5 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

   (a) Suppose that (independently) X_0 is chosen randomly with $P(X_0 = 0) = P(X_0 = 1) = 1/8,\ P(X_0 = 2) = 3/4$. Compute $E(X_3)$.

   (b) Show that the chain is irreducible and solve for the limiting distribution: Solve $\pi = \pi P$ for the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2)$, where $\pi_j > 0,\ j \in \mathcal{S}$, and $\sum_{j \in \mathcal{S}} \pi_j = 1$.

   (c) Compute the variance, given by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n^2 \ - \ \left[\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n \right]^2$$

   (d) Given that the chain is now in state 2, what is the expected amount of time until the chain returns to state 2?

4. Consider a simple random walk with $0 < p < 1$. But now we restrict it to be non-negative in the following way: $P_{0,1} = 1$, and $P_{i,0} = q = 1 - p,\ i \geq 1$; otherwise $P_{i,i+1} = p,\ i \geq 1$ and $P_{i,i-1} = 0,\ i \geq 2$. Thus the state space is still infinite but non-negative; $\mathcal{S} = \{0, 1, 2, \ldots\}$. Imagine that this Markov chain X_n represents the total fortune of a gambler after his n^{th} gamble, where whenever he goes broke, he is given a dollar by a friend so that he can keep gambling. Also note that with probability q, he might go broke after only one gamble no matter what his total fortune is.

   (a) Show that this chain is irreducible.

   (b) Show (via solving $\pi = \pi P$) that it is positive recurrent (for all $0 < p < 1$). HINT: Make sure to use the equation $\sum_{j=0}^{\infty} \pi_j = 1$.

   (c) What is the long-run proportion of time (gambles) that the gambler goes broke?

   (d) Suppose that $p = 0.70$. Given that the gambler has exactly \$5 now, on average how many gambles will it be until he has \$5 again?

5. Consider the simple random walk $\{R_n\}$ with $0 < p < 1$ and in which $p \neq q$. We know that this Markov chain is irreducible and transient (all states are transient). Thus for each state $i \in \mathbb{Z}$, $f_i < 1$ where $f_i =$ the probability that the chain will ever return back to state i given that $R_0 = i$. The objective of this problem is to exactly compute f_i. Note that f_i is the same for all i. So it suffices to derive f_0. So we assume that $R_0 = 0$.

   (a) Letting $M = \max_{n \geq 0} R_n$, and $m = \min_{n \geq 0} R_n$, argue that

$$f_0 = pP(m \leq -1) + qP(M \geq 1).$$

   (b) From (a), solve for f_0 for the two cases $p < q$ and $p > q$.