IEOR 4106, HMWK 4, Professor Sigman

1. Recall how we proved (in lecture) that the simple symmetric random walk is $\{R_n\}$ is *null recurrent* as opposed to positive recurrent (we already proved earlier that it is recurrent as opposed to transient). We proved it in two ways. Here you will prove it by yet a third method. Again we already know and assume that the simple symmetric random walk is recurrent in what follows (e.g., when $p = 1/2$ it is not transient.)

   Recall that by irreducibility, $\{R_n\}$ is positive recurrent if and only if there exists a probability solution to the set of equations $\pi = \pi P$. (Probability solution by definition means that $\pi_j > 0,\ j\ \in \mathbb{Z}$ and $\sum_{j\ \in \mathbb{Z}} \pi_j = 1$. So: Derive the equations $\pi = \pi P$ and show that they reduce to $\pi_{j+1} = \pi_j,\ j\ \in \mathbb{Z}$, and explain why this results in a contradiction that π is a probability solution; hence $\{R_n\}$ must be null recurrent.

   SOLUTION: $\pi = \pi P$ in words is "rate out of state j equals rate into state j" for all states $j \in \mathbb{Z}$. For any $0 < p < 1$, the equations are

$$\pi_j = p\pi_{j-1} + q\pi_{j+1}.$$

   If we focus just on the $j \geq 0$ on the left and sum, we get

$$\sum_{j=0}^{\infty} \pi_j = p\pi_{-1} - q\pi_0 + p\sum_{j=0}^{\infty} \pi_j + q\sum_{j=0}^{\infty} \pi_j.$$

   Since $p + q = 1$ this yields

$$\sum_{j=0}^{\infty} \pi_j = p\pi_{-1} - q\pi_0 + \sum_{j=0}^{\infty} \pi_j.$$

   When $p = q = 1/2$ we thus must have $\pi_{-1} = \pi_0$. Plugging that into the equation $\pi_0 = (1/2)\pi_{-1} + (1/2)\pi_1$ yields $\pi_1 = \pi_0$. Then plugging this into $\pi_1 = (1/2)\pi_0 + (1/2)\pi_2$ yields $\pi_2 = \pi_0$. By induction we get $\pi_j = \pi_0,\ j \geq 0$. (One similarly will get $\pi_j = \pi_0,\ j \leq -1$, but that is not needed in what follows.) Thus if positive recurrent then for some $c > 0$, $\pi_j = c,\ j \geq 0$. But then

$$\sum_{j=0}^{\infty} \pi_j = \sum_{j=0}^{\infty} c = \infty,$$

   which is not possible for positive recurrence; null recurrence follows.

   An even easier way is to simply observe that since (for any $0 < p < 1$) the chain only moves ± 1 it follows that the "rate up from state j to $j+1$ equals the rate down from state $j+1$ to j" for all states $j \in \mathbb{Z}$.

   Writing that out yields
   $p\pi_j = q\pi_{j+1}$ which when $p = 1/2$ yields $\pi_j = \pi_{j+1},\ j \in \mathbb{Z}$.

2. Given a stochastic process $\{X_n : n \geq 0\}$, with discrete state space $\mathcal{S} = \{0, 1\}$, Which of the following random time T are stopping times with respect to $\{X_n\}$ and which are not:

   (a) Let $T_1 = \min\{n \geq 0 : X_n = 0\}$ then define

   $$T = \min\{n > T_1 : X_n = 0\}.$$

   T denotes the *second time* that $\{X_n\}$ visits state 0.

   SOLUTION:

   Yes: if $T = n$, then we know that $X_n = 0$, and $X_k = 0$ for exactly one $0 \leq k < n$; all of this only depends on $\{X_0, \ldots, X_n\}$.

   (b) $T = \min\{n \geq 1 : X_{n-1} = 0, X_n = 1\}$.

   SOLUTION:

   Yes: if $T = 1$ then we know that $(X_0, X_1) = (0, 0)$ and this depends only on $\{X_0, X_1\}$. If $T = n > 1$, then $(X_{n-1}, X_n) = (0, 0)$, and $(X_{k-1}, X_k) \neq (0, 0)$ for any $1 \leq k < n$; all of this only depends on $\{X_0, \ldots, X_n\}$.

   (c) Independent of $\{X_n : n \geq 0\}$, let the sequence $\{U_n : n \geq 0\}$ be iid continuous uniform rvs over the interval $(0, 1)$. $T = \min\{n \geq 0 : U_n > 1/3\}$.

   SOLUTION:

   Yes: We already know that *independent* (of $\{X_n\}$) random times are always stopping times.

   (d) *Continuation:* $T = \min\{n \geq 0 : X_n = 0 \text{ and } U_n > 1/3\}$.

   SOLUTION:

   Yes: $\{T = n\} = \{X_n = 0 \text{ and } U_n > 1/3\}$ which depends *at most* on $\{X_0, \ldots, X_n\}$ from the $\{X_n\}$ sequence; and stopping times are allowed to depend (also) on an entirely independent sequence $\{U_n : n \geq 0\}$.

3. Consider an insurance company that receives claims against it each week, $n \geq 1$, of iid sizes C_n, where the C_n are distributed as a Poisson distribution,

   $$P(C = k) = e^{-\alpha} \frac{\alpha^k}{k!}, \; k \geq 0,$$

   where $\alpha = 20,000$ (dollars).

   Let $T = \min\{n \geq 1 : C_n = 0\}$. Compute

   $$E\Big(\sum_{n=1}^{T} C_n\Big).$$

   SOLUTION:

   $E(C) = \alpha = 20,000$ and since $P(C = 0) = e^{-\alpha}$, we have $P(T = n) = (1 - e^{-\alpha})^{n-1} e^{-\alpha}$, $n \geq 1$ a geometric distribution with $E(T) = e^{\alpha} = e^{20,000} < \infty$. Thus from Wald's equation we obtain the answer as $E(T)E(C) = e^{20,000} \times 20,000$.

4. Consider the Gambler's ruin problem (Markov chain) $\{X_n\}$ with $p = 0.40$, and $N = 10$. Suppose $X_0 = i = 6$. Let $T = \min\{n \geq 1 : X_n \in \{0, 10\}\}$ the time that the gambler stops gambling. Recall that $P_i(N)$ = the probability that the chain hits N before 0 given that $X_0 = i$. Recall that we have exact formulas for these probabilities.

   (a) Compute $E(X_T)$.
   SOLUTION:
   $P(X_T = N) = P_i(N)$, $P(X_T = 0) = (1 - P_i(N))$; thus $E(X_T) = NP_i(N)$, where with $q/p = 3/2$, $i = 6$ and $N = 10$,

   $$P_i(N) = \frac{1 - (3/2)^6}{1 - (3/2)^{10}} = 0.1834.$$

   Thus
   $$E(X_T) = 1.834.$$

   (b) Use (a) together with Wald's equation to find $E(T)$, the expected time until the game ends.
   SOLUTION:
   $X_T = 6 + \sum_{n=1}^{T} \Delta_n$, where the $\{\Delta_n\}$ are iid with $E|\Delta| = 1 < \infty$ and $E(\Delta) = p - q = -0.2$. From Wald's equation we thus obtain $E(X_T) = 6 + E(T)E(\Delta)$ and hence
   $$E(T) = \frac{E(X_T) - 6}{E(\Delta)} = 20.83.$$

   Proving in advance that $E(T) < \infty$ follows easily from finite state space Markov chain theory. For example $E(T) = \sum_{j=1}^{9} S_{6,j} < \infty$ from our transient state analysis ($S = (I - P_T)^{-1}$.)