IEOR 4106, HMWK 4, Professor Sigman

1. Recall how we proved (in lecture) that the simple symmetric random walk is $\{R_n\}$ is *null recurrent* as opposed to positive recurrent (we already proved earlier that it is recurrent as opposed to transient). We proved it in two ways. Here you will prove it by yet a third method. Again we already know and assume that the simple symmetric random walk is recurrent in what follows (e.g., when $p = 1/2$ it is not transient.)

   Recall that by irreducibility, $\{R_n\}$ is positive recurrent if and only if there exists a probability solution to the set of equations $\pi = \pi P$. (Probability solution by definition means that $\pi_j > 0,\ j \in \mathbb{Z}$ and $\sum_{j \in \mathbb{Z}} \pi_j = 1$. So: Derive the equations $\pi = \pi P$ and show that they reduce to $\pi_{j+1} = \pi_j,\ j \in \mathbb{Z}$, and explain why this results in a contradiction that π is a probability solution; hence $\{R_n\}$ must be null recurrent.

2. Given a stochastic process $\{X_n : n \geq 0\}$, with discrete state space $\mathcal{S} = \{0, 1\}$, Which of the following random time T are stopping times with respect to $\{X_n\}$ and which are not:

   (a) Let $T_1 = \min\{n \geq 0 : X_n = 0\}$ then define
   $$T = \min\{n > T_1 : X_n = 0\}.$$
   T denotes the *second time* that $\{X_n\}$ visits state 0.

   (b) $T = \min\{n \geq 1 : X_{n-1} = 0, X_n = 1\}$.

   (c) Independent of $\{X_n : n \geq 0\}$, let the sequence $\{U_n : n \geq 0\}$ be iid continuous uniform rvs over the interval $(0, 1)$. $T = \min\{n \geq 0 : U_n > 1/3\}$.

   (d) *Continuation:* $T = \min\{n \geq 0 : X_n = 0 \text{ and } U_n > 1/3\}$.

3. Consider an insurance company that receives claims against it each week, $n \geq 1$, of iid sizes C_n, where the C_n are distributed as a Poisson distribution,
   $$P(C = k) = e^{-\alpha}\frac{\alpha^k}{k!},\ k \geq 0,$$
   where $\alpha = 20,000$ (dollars).

   Let $T = \min\{n \geq 1 : C_n = 0\}$. Compute
   $$E\Big(\sum_{n=1}^{T} C_n\Big).$$

4. Consider the Gambler's ruin problem (Markov chain) $\{X_n\}$ with $p = 0.40$, and $N = 10$. Suppose $X_0 = i = 6$. Let $T = \min\{n \geq 1 : X_n \in \{0, 10\}\}$ the time that the gambler stops gambling. Recall that $P_i(N) =$ the probability that the chain hits N before 0 given that $X_0 = i$. Recall that we have exact formulas for these probabilities.

   (a) Compute $E(X_T)$.

   (b) Use (a) together with Wald's equation to find $E(T)$, the expected time until the game ends.