1. Recall how we proved (in lecture) that the simple symmetric random walk is \( \{R_n\} \) is null recurrent as opposed to positive recurrent (we already proved earlier that it is recurrent as opposed to transient). We proved it in two ways. Here you will prove it by yet a third method. Again we already know and assume that the simple symmetric random walk is recurrent in what follows (e.g., when \( p = 1/2 \) it is not transient.)

Recall that by irreducibility, \( \{R_n\} \) is positive recurrent if and only if there exists a probability solution to the set of equations \( \pi = \pi P \). (Probability solution by definition means that \( \pi_j > 0, j \in \mathbb{Z} \) and \( \sum_{j \in \mathbb{Z}} \pi_j = 1 \). So: Derive the equations \( \pi = \pi P \) and show that they reduce to \( \pi_{j+1} = \pi_j, j \in \mathbb{Z} \), and explain why this results in a contradiction that \( \pi \) is a probability solution; hence \( \{R_n\} \) must be null recurrent.

2. Given a stochastic process \( \{X_n : n \geq 0\} \), with discrete state space \( S = \{0, 1\} \), Which of the following random time \( T \) are stopping times with respect to \( \{X_n\} \) and which are not:

(a) Let \( T_1 = \min\{n \geq 0 : X_n = 0\} \) then define

\[
T = \min\{n > T_1 : X_n = 0\}.
\]

\( T \) denotes the second time that \( \{X_n\} \) visits state 0.

(b) \( T = \min\{n \geq 1 : X_{n-1} = 0, X_n = 1\} \).

(c) Independent of \( \{X_n : n \geq 0\} \), let the sequence \( \{U_n : n \geq 0\} \) be iid continuous uniform rvs over the interval \((0,1)\). \( T = \min\{n \geq 0 : U_n > 1/3\} \).

(d) Continuation: \( T = \min\{n \geq 0 : X_n = 0 \) and \( U_n > 1/3\} \).

3. Consider an insurance company that receives claims against it each week, \( n \geq 1 \), of iid sizes \( C_n \), where the \( C_n \) are distributed as a Poisson distribution,

\[
P(C = k) = e^{-\alpha} \frac{\alpha^k}{k!}, \quad k \geq 0,
\]

where \( \alpha = 20,000 \) (dollars).

Let \( T = \min\{n \geq 1 : C_n = 0\} \). Compute

\[
E\left(\sum_{n=1}^{T} C_n\right).
\]

4. Consider the Gambler’s ruin problem (Markov chain) \( \{X_n\} \) with \( p = 0.40 \), and \( N = 10 \). Suppose \( X_0 = i = 6 \). Let \( T = \min\{n \geq 1 : X_n \in \{0, 10\}\} \) the time that the gambler stops gambling. Recall that \( P_i(N) = \) the probability that the chain hits \( N \) before 0 given that \( X_0 = i \). Recall that we have exact formulas for these probabilities.

(a) Compute \( E(X_T) \).

(b) Use (a) together with Wald’s equation to find \( E(T) \), the expected time until the game ends.