IEOR 4106, HMWK 5, Professor Sigman

1. Trucks arrive to the GW bridge according to a Poisson process at rate $\lambda_T = 100$ per hour. Independently, cars arrive to the GW bridge according to a Poisson process at rate $\lambda_C = 300$ per hour.

   (a) What is the probability that exactly 75 Trucks arrive during the hour from 1-2PM? **SOLUTION:** By stationary increments, we can use the first hour of time, and we want (since $\lambda_T t = 100(1) = 100$):

   $$P(N_T(1) = 75) = e^{-100}\frac{(100)^{75}}{75!} \approx 0.001499.$$

   (b) What is the probability that exactly 75 Trucks arrive during the hour from 1-2PM given that 102 Trucks already arrived during the earlier hour of 10-11AM?

   SOLUTION: By independent increments, same answer as in (a).

   (c) What is the probability that both 75 Trucks arrive during the hour from 1-2PM and 320 Cars arrive during the hour from 10-11AM?

   SOLUTION: Since the two Poisson processes are independent, we get the product of the two probabilities, where we use stationary increments of each one so as to convert to the first hour of time (as we did in (a)).

   $$P(N_T(1) = 75)P(N_C(1) = 320) = e^{-100}\frac{(100)^{75}}{75!}e^{-300}\frac{(300)^{320}}{320!} \approx 0.00001741.$$

   (d) You go to the GW Bridge at time 12 noon to observe the next arrival. What is the probability that you observe a Truck arrive before a Car? **SOLUTION:** Letting $t_1(T)$ and $t_1(C)$ denote the arrival time of the first Truck and Car, we want

   $$P(t_1(T) < t_1(C)) = \frac{\lambda_T}{\lambda_T + \lambda_C} = 1/4,$$

   where we use the fact that $t_1(T)$ and $t_1(C)$ are independent exponentially distributed random variables, at rates λ_T and λ_C respectively.

   (e) Let *Vehicles* denote Trucks and Cars together (superposition). What is the probability that exactly 1 Vehicle arrives during a given minute of time (1/60 of an hour)?

   SOLUTION: The superposition of the two independent Poisson processes is Poisson at rate $\lambda = \lambda_T + \lambda_C = 400$. We want $P(N(1/60) = 1)$, so we use $\lambda t = (400)(1/60) = 6.67$, and $k = 1$ in the Poisson distribution formula:

   $$P(N(1/60) = 1) = e^{-6.67}\frac{(6.67)^1}{1!} = e^{-6.67}(6.67) \approx 0.00846$$

2. Phone calls come in to your phone according to a Poisson process $\psi = \{t_n : n \geq 1\}$ at rate $\lambda = 5$ per hour. The counting process is given by $\{N(t) : t \geq 0\}$.

   (a) Conditional on $N(1) = 7$, what is the probability that exactly 2 of these 7 arrived in the first 15 minutes of the hour (e.g., during the first 1/4 of the hour)?

SOLUTION: Conditional on $N(1) = 7$, we can represent $(t_1, \ldots, t_7)$ as $(V_{(1)}, \ldots, V_{(7)})$, the order statistics of 7 iid uniform $(0, 1)$ random variables $(V_1, \ldots, V_7)$. The number of $V_{(i)}$ out of 7 that fall $\leq 1/4$ is given by the random variable

$$K = \sum_{i=1}^{7} I\{V_{(i)} \leq 1/4\} = \sum_{i=1}^{7} I\{V_i \leq 1/4\},$$

and hence has a binomial $(7, p)$ distribution with $p = P(V \leq 1/4) = 1/4$. Thus the answer is $P(K = 2) = \binom{7}{2}(1/4)^2(3/4)^5$.

(b) Explain why the above answer would remain the same if $N(1)$ is replace by $N(9) - N(8)$.
SOLUTION: By stationary increments, $N(9) - N(8)$ has the same distribution as $N(1)$.

(c) Suppose that the Poisson process $\psi = \{t_n : n \geq 1\}$ is actually the independent superposition of two Poisson processes: $\psi_1 = \{t_n(1) : n \geq 1\}$, a Poisson process at rate $\lambda_1 = 3$, with counting process $\{N_1(t) : t \geq 0\}$ (DOMESTIC CALLS), and $\psi_2 = \{t_n(2) : n \geq 1\}$, a Poisson process at rate $\lambda_2 = 2$, with counting process $\{N_2(t) : t \geq 0\}$ (FOREIGN CALLS).

i. What is the probability that the first 4 calls you get are all Domestic?
SOLUTION: Each call is, independently, Domestic with probability $p = \lambda_1/(\lambda_1 + \lambda_2) = 3/5$, and Foreign with probability $q = 1 - p = \lambda_2/(\lambda_1 + \lambda_2) = 2/5$. Thus the answer is $p^4 = (3/5)^4$.

ii. Let $K =$ the number of calls until you get the first Foreign call. Find $E(K)$.
SOLUTION:
K has a geometric distribution with probability of success $p = 2/5$; $P(K = n) = (3/5)^{n-1}(2/5),\ n \geq 1$. Thus $E(K) = 1/p = 5/2$.

iii. Conditional on $N(1) = 7$, what is the probability that exactly 2 of these 7 are Foreign and 5 are Domestic?
SOLUTION:
The number out of these 7 that are Foreign is given by a random variable F that has a binomial $(7, 2/5)$ distribution, thus $P(F = 2) = \binom{7}{2}(2/5)^2(3/5)^5$. One could also use a binomial $(7, 3/5)$ rv D denoting the number out of 7 that are Domestic, and get the same answer via $P(D = 5) = \binom{7}{5}(3/5)^5(2/5)^2$.

3. According to a Poisson process at rate $\lambda = 20$ per day, a company buys units (100 share blocks) of stock A and holds on to each unit, independently of other units, for H days, where H has an exponential distribution with $E(H) = 60$ (days).

Assume that initially (time $t = 0$) no units of stock A are held.

(a) Compute the expected number of units held at times $t = 20$, $t = 50$ and $t = 90$ days.
SOLUTION: This is an example of a M/G/∞ queue, with "service time" distribution $P(H > s) = e^{-\mu s}$ where $\mu = 1/60$. Let $X(t)$ denote the number of units held at time t. We know that $X(t)$ has a Poisson distribution with mean $\alpha(t) = E(X(t))$ (expected value) given by

$$\alpha(t) = \lambda \int_0^t P(H > s)ds = \rho(1 - e^{-\mu t}),$$

where $\rho = \lambda/\mu = 1200$. Thus $\alpha(t) = 1200(1 - e^{-t/60})$.
$\alpha(20) = 340.16,\ \alpha(50) = 678.48,\ \alpha(90) = 932.24.$

(b) Repeat (a), (b) when H has a uniform distribution on $(0, 120)$.
SOLUTION:

$$P(H > s) = \begin{cases} \frac{120-s}{120} & \text{if } s \in (0, 120), \\ 0 & \text{if } s \geq 120, \end{cases}$$

$$\alpha(t) = \lambda \int_0^t P(H > s)ds = 20 \int_0^t \frac{120 - s}{120} ds, \ t \in (0, 120),$$

so we have to use the above integration formula for our given times; $\alpha(20), \alpha(50), \ \alpha(90)$.
$\alpha(20) = 366.67, \ \alpha(50) = 791.67, \ \alpha(90) = 1125$.
Of course once $t = 120$, then since $P(H > s) = 0, \ s \geq 120$, we would get

$$\alpha(t) = \lambda \int_0^\infty P(H > s)ds = \lambda E(S) = \rho = 1200, \ t \geq 120.$$

(c) Repeat (a), (b) when H has a uniform distribution on $(40, 80)$.
SOLUTION:

$$P(H > s) = \begin{cases} 1 & \text{if } s \in [0, 40), \\ \frac{80-s}{40} & \text{if } s \in [40, 80), \\ 0 & \text{if } s \geq 80, \end{cases}$$

So: In this case we must pay attention to the fact that the tail $P(H > t) = 1, \ t \in [0, 40)$, so that

$$\alpha(t) = \lambda \int_0^t ds = \lambda t, \ t \in [0, 40),$$

and then for any $t \in [40, 80)$, since $\int_0^t = \int_0^{40} + \int_{40}^t$,

$$\alpha(t) = 40\lambda + \lambda \int_{40}^t \frac{80 - s}{40} ds,$$

and finally, $\alpha(t) = \rho, \ t \geq 80$. $\rho = \lambda E(H) = (20)(60) = 1200$.
Thus $\alpha(20) = 20\lambda = (20)(20) = 400$
$\alpha(50) = \alpha(40) + \int_{40}^{50} \frac{80-s}{40} ds = 400 + 2 \int_{40}^{50} \frac{80-s}{40} ds = 975$.
$\alpha(90) = \rho = (20)(60) = 1200$.

(d) As $t \to \infty$, what is the limiting probability distribution for the number of units held? Does it depend on which of the 3 distributions above are used for holding times?
SOLUTION:
In all 3 cases, $E(H) = 60$ is the same, and hence so is $\rho = \lambda E(H) = (20)(60) = 1200$. The limiting distribution (for an $M/G/\infty$ queue) is always Poisson with mean ρ; hence is the same for all 3 cases.