# IEOR 4106, HMWK 6, Professor Sigman

1. *Printer with jams*: Jobs arrive to a computer printer according to a Poisson process at rate $\lambda$. Jobs are printed one at a time requiring iid printing times that are exponentially distributed with rate $\mu$. Jobs wait in a FIFO queue before entering service.

   Additionally, independently, the printer jams at times that form a Poisson process at rate $\gamma$. Whenever a jam occurs the job being printed (if any) is removed (and lost), and the printer continues printing the remaining jobs. If the printer has no jobs, then the jam has no effect (e.g., the printer instantly resets). Let $X(t)$ denote the number of jobs at the printer at time $t$.

   (a) Suppose that right now a job is in the midst of being processed. Let $T$ denote how long it will be (from now) until either the job is complete or lost (if so). Argue that $T \sim exp(\mu + \gamma)$.
   **SOLUTION:** By the memoryless property of the exponential service times and interjam times, $T = \min\{S, J\}$, where $S \sim exp(\mu)$ and $J \sim exp(\gamma)$ and they are independent. Thus $T \sim exp(\mu + \gamma)$.

   (b) Argue that $\{X(t)\}$ is a Birth and Death process; give the birth rates $\{\lambda_i\}$ and the death rates $\{\mu_i\}$. **SOLUTION:** By the memoryless property of service times (S), interarrival times (A), and interjam times (J), we have (at most) at any given time $t$ 3 rvs competing to be the minimum to determine what happens next: $S, J, A$. When $X(t) = 0$ only $A$ is next, when $X(t) \geq 1$ we have all 3, and independent of the past in all cases. So we do have a CTMC. And since either there is an arrival (birth) or a departure (death, due to either service completion or jam), it is a Birth and Death process.

   (c) Give the holding time rates $\{a_i : i \geq 0\}$, and the transition probabilities $P_{i,j}$ for the embedded discrete-time Markov chain.
   **SOLUTION:** $a_0 = \lambda,\ a_i = \lambda + \mu + \gamma,\ i \geq 1$.
   $P_{0,1} = 1$ and otherwise $P_{i,i+1} = \lambda/(\lambda + \mu + \gamma),\ P_{i,i-1} = (\mu + \gamma)/(\lambda + \mu + \gamma),\ i \geq 1$.

   (d) Explain how in fact this chain is the same as for a regular FIFO M/M/1 (but with modified service rate given as....).
   **SOLUTION:** The Birth and Death balance equations are:

   $$\lambda P_j = (\mu + \gamma)P_{j+1},\ j \geq 0.$$

   If we define $\overline{\mu} = \mu + \gamma$, then we can re-write as

   $$\lambda P_j = \overline{\mu} P_{j+1},\ j \geq 0.$$

   This is exactly the same as for a M/M/1 queue with arrival rate $\lambda$ and service rate $\overline{\mu}$ so we must have $\lambda < \overline{\mu} = \mu + \gamma$ for the solution to exist in which case it is geometric: Letting $\overline{\rho} = \lambda/\overline{\mu}$,
   $P_j = \overline{\rho}^j(1 - \overline{\rho}),\ j \geq 0$.

   (e) What is the long-run proportion of jam times that are effective (e.g., remove a job).
   **SOLUTION:** A jam time removes a job if and only if it finds the system non-empty. By PASTA, the long-run proportion of jam times that find the system non-empty is the same as the long-run proportion of time that the system is non-empty, $1 - P_0 = \overline{\rho}$.

2. Consider 5 iPhones, each independently having a battery lifetime that is exponentially distributed with mean 2 years. Once a battery breaks down, the iPhone immediately goes to a facility to have the battery replaced. The replacing facility handles only the above 5 phones (no others), but can only work at most on 2 phones at a time (the others wait in queue (line); the replacing facility is a $2-$ server in parallel system; like a FIFO G/M/2 queue but in which the arrivals are the down machines "arriving" for repair). Replacing times are exponentially distributed with mean 0.3 year (hence rate $\lambda = 10/3$.). Let $X(t)$ denote the number of iPhones at time $t$ that have working batteries.

   (a) Argue that $\{X(t)\}$ forms a continuous-time Markov chain. Give the holding time rates $a_0, a_1, a_2, a_3, a_4, a_5$, and the transition probabilities $P_{i,j}$ for the embedded discrete-time Markov chain.
   **SOLUTION:** $a_0 = 2\lambda,\ a_1 = 2\lambda+\mu,\ a_2 = 2\lambda+2\mu,\ a_3 = 2\lambda+3\mu,\ a_4 = \lambda+4\mu,\ a_5 = 5\mu$.

$$
\begin{aligned}
P_{0,1} &= 1 \\
P_{1,2} &= 2\lambda/(2\lambda+\mu) \\
P_{1,0} &= \mu/(2\lambda+\mu) \\
P_{2,3} &= 2\lambda/(2\lambda+2\mu) \\
P_{2,1} &= 2\mu/(2\lambda+2\mu) \\
P_{3,4} &= 2\lambda/(2\lambda+3\mu) \\
P_{3,2} &= 3\mu/(2\lambda+3\mu) \\
P_{4,5} &= \lambda/(\lambda+4\mu) \\
P_{4,3} &= 4\mu/(\lambda+4\mu) \\
P_{5,4} &= 1.
\end{aligned}
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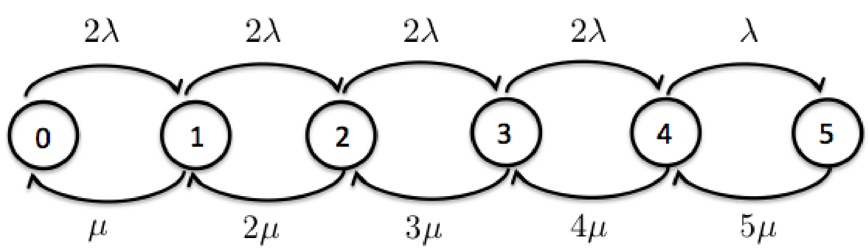
   (b) Draw the rate diagram.
   **SOLUTION:**




   (c) Explain why $\{X(t)\}$ is a Birth and Death process, and give the birth and death rates.
   **SOLUTION:** Only jumps of size $\pm 1$ occur; a battery breaks down $(-1)$, a battery is repaired $(+1)$.
   $\lambda_0 = 2\lambda,\ \lambda_1 = 2\lambda,\ \lambda_2 = 2\lambda,\ \lambda_3 = 2\lambda\ \lambda_4 = \lambda,\ \ (\lambda_5 = 0$ since there are only 5 batteries.).
   $\mu_0 = 0,\ \mu_1 = \mu,\ \mu_2 = 2\mu,\ \mu_3 = 3\mu,\ \mu_4 = 4\mu,\ \mu_5 = 5\mu$.

(d) Solve for the limiting (stationary) distribution $(P_0, \ldots, P_5)$. **SOLUTION:**

$$\begin{aligned}
2\lambda P_0 &= \mu P_1 && (1)\\
2\lambda P_1 &= 2\mu P_2 && (2)\\
2\lambda P_2 &= 3\mu P_3 && (3)\\
2\lambda P_3 &= 4\mu P_4 && (4)\\
\lambda P_4 &= 5\mu P_5 && (5)
\end{aligned}$$

With $\lambda = 10/3$, $\mu = 2$ and $\sum_{j=0}^{5} P_j = 1$, we get the solution as follows:

$$\begin{aligned}
P_0 &= \frac{729}{16709} = 0.00028\\
P_1 &= \frac{2430}{16709} = 0.00373\\
P_2 &= \frac{4050}{16709} = 0.02489\\
P_3 &= \frac{4500}{16709} = 0.11063\\
P_4 &= \frac{3750}{16709} = 0.36877\\
P_5 &= \frac{1250}{16709} = 0.49169
\end{aligned}$$

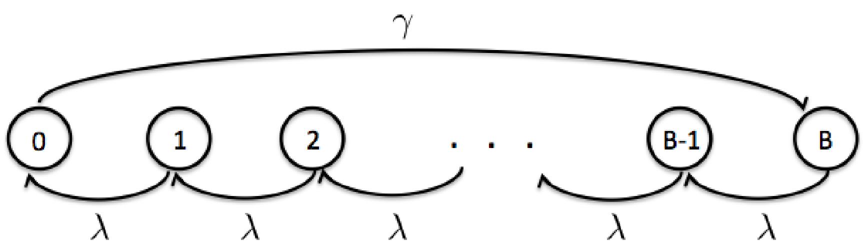
(e) Compute the average number of iPhones with working batteries.
**SOLUTION:**

$$\sum_{j=0}^{5} jP_j = 0.00373 + 2 \times 0.00373 + 3 \times 0.11063 + 4 \times 0.36877 + 5 \times 0.49169 = 4.27661$$

3. *Inventory model I*: A retailer sells headphones one at a time according to demand which forms a Poisson process at rate $\lambda$: At Poisson arrival time $t_n$ ($n^{th}$ demand request), the inventory drops by 1 if the inventory is non-empty. If the inventory is empty at a request time, then nothing happens, that demand request is "lost". The amount in inventory starts off as $B \geq 2$. As soon as the Inventory drops down to 0, it will be re-stocked up to $B$ after an exponential amount of time $L$ (lead time) at rate $\gamma$, independent of the past. Again: during those $L$ time units, all demand is lost. Let $X(t)$ denote the inventory level at time $t$. The state space is thus $\{0, 1, \ldots, B\}$.

(a) Argue that $\{X(t)\}$ forms a CTMC, and find both the holding time rates $a_j$ and the embedded MC transition matrix $P = (P_{i,j})$.
**SOLUTION:**
Given that $X(t) = i > 0$, then the chain will after an exponential $\lambda$ amount of time, independent of the past (via memoryless property of the Poisson process), change to state $i - 1$. Given that $X(t) = 0$, then the chain will after an exponential $\gamma$ amount of time, independent of the past (via memoryless property of the exponential lead time), and independent of the future Poisson process of demand, change to state $B$. Thus, $\{X(t) : t \geq 0\}$ forms a CTMC.
$a_0 = \gamma$, while $a_j = \lambda,\ 1 \leq j \leq B$. $P_{0,B} = 1$, while $P_{i,i-1} = 1,\ 1 \leq i \leq B$.

(b) Explain why it is not a birth and death process. Draw the rate diagram.
**SOLUTION:** $P_{0,B} = 1$ where $B \geq 2$, so a jump of size $B > 1$ happens next whenever the system empties instead of only jumps of size 1 as is required to be a birth and death process.




(c) Set up the balance equations: "rate out of state $j$ equals rate into state $j$" for all $j \in \mathcal{S}$.
**SOLUTION:**

$$\begin{aligned} \gamma P_0 &= \lambda P_1 && (6) \\ \lambda P_1 &= \lambda P_2 && (7) \\ &\vdots && (8) \\ \lambda P_{B-1} &= \lambda P_B && (9) \\ \lambda P_B &= \gamma P_0. && (10) \end{aligned}$$

(d) Solve the balance equations.
**SOLUTION:**
It is immediate that $P_1 = (\gamma/\lambda)P_0$ and that

$$P_1 = P_2 = \cdots = P_B,$$

hence

$$P_1 = P_2 = \cdots = P_B = (\gamma/\lambda)P_0.$$

Thus using

$$\sum_{j=0}^{B} P_j = 1,$$

yields that

$$P_0(1 + B(\gamma/\lambda)) = 1,$$

or

$$P_0 = (1 + B(\gamma/\lambda))^{-1},$$

and

$$P_1 = P_2 = \cdots = P_B = (\gamma/\lambda)(1 + B(\gamma/\lambda))^{-1}.$$

(e) Find the long-run average amount of inventory that the retailer has.
**SOLUTION:**

$$\sum_{j=0}^{B} jP_j = \sum_{j=1}^{B} jP_j = (\gamma/\lambda)(1+B(\gamma/\lambda))^{-1}\sum_{j=1}^{B} j$$
$$= (\gamma/\lambda)(1+B(\gamma/\lambda))^{-1}\,((B)(B+1)/2)$$
$$= \frac{1}{B+\frac{\lambda}{\gamma}}\,((B)(B+1)/2)\,.$$

(f) What is the long-run proportion of demand requests that are lost?
**SOLUTION:**
This is the same as the long-run proportion of demand requests that find the inventory empty. From PASTA, this is equal to the long-run *proportion of time* that the inventory is empty: $P_0 = (1+B(\gamma/\lambda))^{-1}$.

(g) If each headphone set costs the retailer \$$c$, but sells for \$$2c$,then what is the long-run rate at which the retailer earns money? (e.g., how much money per unit time).
**SOLUTION:**
$\lambda$ is the rate at which requests come in, and each time they do so a profit of \$$c$ is earned as long as the inventory is not empty when the request comes in; hence we want
$$c\lambda(1-\pi_0^a),$$
where $\pi_0^a$ denotes the proportion (fraction) of (the Poisson) request times that find the inventory empty. From PASTA, $\pi_0^a = P_0$ and hence we want
$$c\lambda(1-P_0) = c\lambda(1-(1+B(\gamma/\lambda))^{-1}).$$

4. For the rat in the closed maze, with (general) holding time rates $a_i > 0,\ 1 \le i \le 4$, draw the rate diagram, and set up the balance equations.

**SOLUTION:**

Balance Equations for general values of the $a_i$:

$$a_1P_1 = a_2P_2(1/2) + a_3P_3(1/2) \tag{11}$$
$$a_2P_2 = a_1P_1(1/2) + a_4P_4(1/2) \tag{12}$$
$$a_3P_3 = a_1P_1(1/2) + a_4P_4(1/2) \tag{13}$$
$$a_4P_4 = a_2P_2(1/2) + a_3P_3(1/2). \tag{14}$$

(a) Show that $a_1P_1 = a_2P_2 = a_3P_3 = a_4P_4$.
**SOLUTION:** Fom the above balance equations, the right hand side of equations 1 and 4 are identical hence $a_1P_1 = a_4P_4$, similarly for rows 2 and 3 yielding $a_2P_2 = a_3P_3$. Now replacing $a_3P_3$ by $a_2P_2$ in equation 1 yields $a_1P_1 = a_2P_2(1/2) + a_2P_2(1/2) = a_2P_2$. Thus $a_1P_1 = a_2P_2 = a_3P_3 = a_4P_4$.

(b) Solve the balance equations.
**SOLUTION:** Using $P_1+P_2+P_3+P_4 = 1$ together with (a), Solve $P_1(1+(a_1/a_2)+(a_1/a_3)+(a_1/a_4)) = 1$ for $P_1$:
$$P_1 = (1+(a_1/a_2)+(a_1/a_3)+(a_1/a_4))^{-1}.$$

$$
\begin{aligned}
P_2 &= (a_1/a_2)P_1 && (15) \\
P_3 &= (a_1/a_3)P_1 && (16) \\
P_4 &= (a_1/a_4)P_1. && (17)
\end{aligned}
$$