IEOR 4106, HMWK 6, Professor Sigman

1. *Printer with jams*: Jobs arrive to a computer printer according to a Poisson process at rate λ. Jobs are printed one at a time requiring iid printing times that are exponentially distributed with rate μ. Jobs wait in a FIFO queue before entering service.

   Additionally, independently, the printer jams at times that form a Poisson process at rate γ. Whenever a jam occurs the job being printed (if any) is removed (and lost), and the printer continues printing the remaining jobs. If the printer has no jobs, then the jam has no effect (e.g., the printer instantly resets). Let $X(t)$ denote the number of jobs at the printer at time t.

   (a) Suppose that right now a job is in the midst of being processed. Let T denote how long it will be (from now) until either the job is complete or lost (if so). Argue that $T \sim exp(\mu + \gamma)$.

   (b) Argue that $\{X(t)\}$ is a Birth and Death process; give the birth rates $\{\lambda_i\}$ and the death rates $\{\mu_i\}$.

   (c) Give the holding time rates $\{a_i : i \geq 0\}$, and the transition probabilities $P_{i,j}$ for the embedded discrete-time Markov chain.

   (d) Explain how in fact this chain is the same as for a regular FIFO M/M/1 (but with modified service rate given as....).

   (e) What is the long-run proportion of jam times that are effective (e.g., remove a job).

2. Consider 5 iPhones, each independently having a battery lifetime that is exponentially distributed with mean 2 years. Once a battery breaks down, the iPhone immediately goes to a facility to have the battery replaced. The replacing facility handles only the above 5 phones (no others), but can only work at most on 2 phones at a time (the others wait in queue (line); the replacing facility is a $2-$ server in parallel system; like a FIFO G/M/2 queue but in which the arrivals are the down machines "arriving" for repair). Replacing times are exponentially distributed with mean 0.3 year (hence rate $\lambda = 10/3$.). Let $X(t)$ denote the number of iPhones at time t that have working batteries.

   (a) Argue that $\{X(t)\}$ forms a continuous-time Markov chain. Give the holding time rates $a_0, a_1, a_2, a_3, a_4, a_5$, and the transition probabilities $P_{i,j}$ for the embedded discrete-time Markov chain.

   (b) Draw the rate diagram.

   (c) Explain why $\{X(t)\}$ is a Birth and Death process, and give the birth and death rates.

   (d) Solve for the limiting (stationary) distribution $(P_0, \ldots, P_5)$.

   (e) Compute the average number of iPhones with working batteries.

3. *Inventory model I*: A retailer sells headphones one at a time according to demand which forms a Poisson process at rate λ: At Poisson arrival time t_n (n^{th} demand request), the inventory drops by 1 if the inventory is non-empty. If the inventory is empty at a request time, then nothing happens, that demand request is "lost". The amount in inventory starts off as $B \geq 2$. As soon as the Inventory drops down to 0, it will be re-stocked up to B after an exponential amount of time L (lead time) at rate γ, independent of the past. Again: during those L time units, all demand is lost. Let $X(t)$ denote the inventory level at time t. The state space is thus $\{0, 1, \ldots, B\}$.

(a) Argue that $\{X(t)\}$ forms a CTMC, and find both the holding time rates a_j and the embedded MC transition matrix $P = (P_{i,j})$.

(b) Explain why $\{X(t)\}$ is not a birth and death process. Draw a rate diagram.

(c) Set up the balance equations: "rate out of state j equals rate into state j" for all $j \in \mathcal{S}$.

(d) Solve the balance equations.

(e) Find the long-run average amount of inventory that the retailer has.

(f) What is the long-run proportion of demand requests that are lost? (HINT: PASTA)

(g) If each headphone set costs the retailer \$$c$, but sells for \$$2c$, then what is the long-run rate at which the retailer earns money? (e.g., how much money per unit time).

4. For the rat in the closed maze, with (general) holding time rates $a_i > 0,\ 1 \le i \le 4$, draw the rate diagram, and set up the balance equations.

   (a) Show that $a_1P_1 = a_2P_2 = a_3P_3 = a_4P_4$.

   (b) Solve the balance equations.