IEOR 4106, HMWK 7, Professor Sigman

1. For generations, a family (and their descendants) keep a Galapagos tortoise in their yard as a pet. As soon as it dies, they immediately get a new just born one. Suppose that the average lifetime of such a tortoise is 250 years, and the *standard deviation* is $\sigma = \sqrt{\sigma^2} = 30$. Suppose (way out in the infinite future) you visit the family. What is the expected *total lifetime* of the tortoise you find (e.g., sum of age plus remaining lifetime)? Is the inspection paradox holding here?

   SOLUTION:

   We need to compute average *spread*;

   $$\frac{E(X^2)}{E(X)}.$$

   We compute as follows: We are told that $E(X) = 250$. Also, $E(X^2) = \sigma^2 + E^2(X) = 30^2 + (250)^2 = 63400$. Thus

   $$\frac{E(X^2)}{E(X)} = 253.6$$

   Notice how the inspection paradox is here: $253.6 > 250 = E(X)$.

2. *Printer with disasters*: Jobs arrive to a computer printer according to a Poisson process at rate λ. Jobs are printed one at a time requiring iid printing times that are exponentially distributed with rate μ. Jobs wait in a FIFO queue before entering service.

   Additionally, independently, disasters occur according to a Poisson process at rate γ. Whenever a disaster occurs all jobs are removed and lost (both any in line and the one in service). If the printer has no jobs, then the disaster has no effect (e.g., the printer instantly resets, waiting for new arrivals). Let $X(t)$ denote the number of jobs at the printer at time t.

   (a) $\{X(t)\}$ is not a Birth and Death process (why?). But it is an irreducible CTMC. Draw the rate diagram. We will show below that the chain is always positive recurrent.

   SOLUTION:

   No, it is not a B&D process, because the state jumps to 0 (at rate γ) from any state $j \geq 1$ when a disaster occurs.

   (b) Set up the balance equations, with the first one as

   $$\lambda P_0 = \mu P_1 + \gamma(1 - P_0).$$

   Explain why this indeed is the first equation: "Rate out of state 0 equals rate into state 0". (You are not expected to solve these equations.......very difficult.....)

   SOLUTION:

   Only a birth (arrival) can take the process out of state 0. To get into state 0 there are two ways: either from a disaster which, at rate γ takes the process there from any state $j > 0$, plus if the process is in state $j = 1$ and a death (service completion) occurs, rate μ. $1 - P_0 = \sum_{j=1}^{\infty} P_j$ = the proportion of time the system is non-empty, e.g., for which $X(t) > 0$. This gives us the first equation.

The other equations are given by

$$(\lambda + \mu + \gamma)P_j = \lambda P_{j-1} + \mu P_{j+1},\ j \geq 1.$$

If $X(t) = j \geq 1$, all 3 possibilities (birth, death, disaster) can occur to take the process out of state j, hence the holding time rate $a_j = \lambda + \mu + \gamma$. Getting back into state $j \geq 1$ can only come from a birth or a death as in a standard $M/M/1$ queue.

(c) Explain why $E(T_{0,0}) \leq 1/\gamma < \infty$, hence the chain is always positive recurrent.
SOLUTION: Since the chain is irreducible, all states together are either positive reccurrent, null recurrent or transient. To show positive recurrence, it thus suffices to show that state 0 is positive recurrent. The chain can become empty in two possible ways: (1) from any disaster, and (2) from a service completion if $X(t) = 1$. Thus the time until the chain re-enters state 0 if $X(0) = 0$ is $\leq$ the time until the next disaster, which we will denote that by $t_D \sim expo(\gamma)$. (Even if T_D is a remaining disaster time, it is still $exp(\gamma)$ by the memoryless property.) Thus $E(T_{0,0}) \leq E(T_D) \leq 1/\gamma < \infty$.

3. Consider the M/M/1 queue (arrival rate λ service time rate μ) with impatient customers: Each customer independently will get impatient after an amount of time that is exponentially distributed at rate γ while waiting in line (queue) and leave before ever entering service, and without ever returning. A customer who does enter service completes service (e.g., customers are only impatient while waiting in the line, not when in service.)

(a) Set up the birth and death balance equations but do not try to solve in general.
SOLUTION:

$$\lambda P_j = (\mu + j\gamma)P_{j+1},\ j \geq 0.$$

The $j\gamma$ on the right hand side represents the rate at which, when $X(t) = j + 1$, the chain has a death due to impatience: all j waiting in the line (queue) are competing to cause the next death; e.g., the minimum of j iid exponentials at rate γ is exponential at rate $j\gamma$. Only the customer in service can cause a death at rate μ.

(b) Show that in the special case when $\gamma = \mu$, you can solve the birth and death balance equations that you set up in (a); solve them. What famous other CTMC model has these same birth and death balance equations?
SOLUTION: We get

$$\lambda P_j = (j + 1)\mu P_{j+1},\ j \geq 0,$$

which are the same equations as for the $M/M/\infty$ model; the solution is thus Poisson with mean $\rho = \lambda/\mu$:

$$P_j = e^{-\rho}\frac{\rho^j}{j!},\ j \geq 0.$$

4. Consider c (identical) ATM machines working in parallel with one FIFO line (queue). Customers arrive at an existing rate $\lambda < \infty$, and service times $\{S_n\}$ have an existing average of $1/\mu < \infty$. (Assume that $\lambda < c\mu$.) Letting the "system" be just the set of c servers, give an expression for the long-run average number of busy servers: If $Y(t) =$ the number of busy servers at time t, we want

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(s)ds.$$

SOLUTION:

$l = \lambda w$ applied to the set of c servers as our "system". Here, sojourn times $W_j = S_j$, are the service times; $l = \lambda(1/\mu) = \rho$.

5. Consider a renewal process with iid interarrival times $\{X_n\}$ with finite (and non-zero) moments $E(X), E(X^2), E(X^3)$. Let $A(t)$ denote the forward recurrence time, $A(t) = t_{N(t)+1} - t,\ t \geq 0$.

   (a) Graph $\{A^2(t) : t \geq 0\}$.
   SOLUTION: Recalling the right triangles that one gets by graphing $\{A(t) : t \geq 0\}$, we now get the square of each: The first "cycle" is given by $\{A^2(s) : 0 \leq s < X_1\} = \{(X_1 - s)^2 : 0 \leq s < X_1\}$, then is followed by iid such similar cycles over $X_2, X_3, \ldots$.

   (b) Compute (wp1):

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A^2(s)ds.$$

   SOLUTION:
   We can use the Renewal Reward Theorem to express the answer as

$$\frac{E(R)}{E(X)},$$

   where we let $A^2(s)$ denote the rate at which money is earned at time s. The reward over the first cycle is thus

$$R = R_1 = \int_0^{X_1} A^2(s)ds = \int_0^{X_1} (X_1 - s)^2 ds = \int_0^{X_1} s^2 ds = \frac{X_1^3}{3}.$$

$$\frac{E(R)}{E(X)} = \frac{E(X^3)}{3E(X)}.$$

6. Recall the Inventory Model that you previously modeled as a CTMC in HMWK 8: A retailer sells headphones one at a time according to demand which forms a Poisson process at rate λ: At Poisson arrival time t_n (n^{th} demand request), the inventory drops by 1 if the inventory is non-empty. If the inventory is empty at a request time, then nothing happens, that demand request is "lost". The amount in inventory starts off as $B \geq 2$. As soon as the Inventory drops down to 0, it will be re-stocked up to B after an exponential amount of time L (lead time) at rate γ, independent of the past. Again: during those L time units, all demand is lost. Let $X(t)$ denote the inventory level at time t. The state space is thus $\{0, 1, \ldots, B\}$.

   Here we will use the Renewal Reward Theorem to re-derive the limiting probabilities $\{P_i : 0 \leq i \leq B\}$ and even allow for a general distribution of lead time L (does not have to be exponentially distributed.)

   (a) With $X(0) = B$, let $s_1 =$ time until the inventory returns back to level B for the first time, let $s_2 =$ the time until the inventory returns back to level B for the second time, and in general, let $s_n =$ the time until the inventory returns back to level B for the n^{th} time, $n \geq 1$. Note that $s_1 = t_B + L$. Argue that $\{s_n : n \geq 1\}$ forms a renewal point process, that is, letting the cycle lengths be denoted by $X_n = s_n - s_{n-1},\ n \geq 1$ (with $s_0 \stackrel{\text{def}}{=} 0$), they are iid distributed as $X = X_1 = s_1 = t_B + L$.

SOLUTION:
Whenever the inventory enters state B at one of the times s_n, the Poisson process starts over again and is independent of its past via the memoryless property and is independent of the past iid lead times. Thus $X_{n+1} = s_{n+1} - s_n$ has the same distribution as X_n and is independent of $X_1, \ldots, X_n$. Thus indeed the $\{X_n\}$ are iid. The expected cycle length is $E(X) = E(t_B + L) = B/\lambda + 1/\gamma$.

(b) Let

$$P_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t I\{X(s) = i\}ds,$$

denote the long-run proportion of time that the inventory is at level i, $0 \le i \le B$. (These were computed by solving the balance equations in HMWK 8; we will compute differently now.) Letting for a given fixed i

$$R = R_1 = \int_0^{X_1} I\{X(s) = i\}ds,$$

(the total amount of time spent in state i during the first cycle) denote the "reward over the first cycle", argue that $E(R) = 1/\lambda,\ 1 \le i \le B$ and $E(R) = 1/\gamma,\ i = 0$. (The point is that we can imagine that rewards are collected continuously at rate 1 whenever $X(t) = i$ and at rate 0 otherwise.) More generally, argue that for $j \ge 1$

$$R_j = \int_{s_{j-1}}^{s_j} I\{X(s) = i\},$$

are iid rewards over the j^{th} cycle, and that $\{(X_j, R_j)\}$ are iid.

SOLUTION: For $1 \le i \le B$, $R_1 = \int_0^{X_1} I\{X(s) = i\}ds$ = the amount of time during a cycle that the inventory remains in state i before dropping to $i-1$. That time is an interarrival time T from the Poisson process of demands, and hence $E(R) = E(T) = 1/\lambda$, after the amount T, the inventory drops to level $i-1$.

For $i = 0$, the amount of time spent in state 0 during a cycle is a lead time L, and hence $E(R) = E(L) = 1/\gamma$. After L, the inventory is up at B again.

Since the cycles are iid, so are the $\{(X_j, R_j)\}$.

(c) Letting

$$R(t) = \int_0^t I\{X(s) = i\}ds,$$

denote the total reward collected up until time t, derive an explicit expression for each P_j , using renewal reward, $P_j = \lim_{t\to\infty} R(t)/t = E(R)/E(X)$. (You should get exactly the same answer as when you solved the balance equations in HMWK 8.)

SOLUTION:
We have the answer as $E(R)/E(X)$, where $E(X) = E(t_B + L) = B/\lambda + 1/\gamma$, and $E(R) = 1/\lambda$ when $i \ne 0$ and $E(R) = 1/\gamma$ when $i = 0$. Thus
$P_1 = \ldots = P_B = (1/\lambda)/(B/\lambda + 1/\gamma)$, and $P_0 = (1/\gamma)/(B/\lambda + 1/\gamma)$. Multiplying both the numerator and denominator of each by γ yields the same form of the solution as from HMWK 8: $P_0 = (1/(1 + B(\gamma/\lambda)),\ P_j = (\gamma/\lambda)P_0,\ 1 \le j \le B$.

(d) Suppose that we change the model to allow the iid lead times $L > 0$ to have a general distribution $G(x) = P(L \le x),\ x \ge 0$, with $E(L) = 1/\gamma < \infty$. In other words we do not assume they are exponentially distributed anymore. Explain why (in general now) $\{X(t) : t \geq 0\}$ is no longer a CTMC, but that the $\{P_j\}$ defined above still exist and are exactly the same value as when the L are exponentially distributed.

SOLUTION: The memoryless property that we had for lead times L is now no longer there; if for example $X(t) = 0$, then we can not say that the remaining lead time is independent of the past, nor would we even know its distribution; it would depend on the time t as well as the past.

But: $\{(X_j, R_j)\}$ are still iid and the values $E(X)$ and $E(R)$ are the same as before, so the renewal reward theorem yields the same answer as before.

SIDE NOTE: We could also obtain

$$\sum_{j=0}^{B} jP_j = \sum_{j=1}^{B} jP_j = \frac{1}{B + \frac{\lambda}{\gamma}} \left((B)(B+1)/2\right),$$

directly by using $l = \lambda w$: The rate at which items arrive to the inventory is given by B items every $E(X)$ amount of time, hence by renewal reward, the rate is

$$\overline{\lambda} = \frac{B}{E(X)} = \frac{B}{\frac{B}{\lambda} + \frac{1}{\gamma}}.$$

Meanwhile, the average sojourn time w of an item is derived as follows:

Of the B items that arrive all at once at the beginning of a cycle, letting T_i denote iid exponential λ interarrival times of demand, the first waits $W_1 = T_1$, the second waits $W_2 = T_1 + T_2$, and so on, finally the B^{th} waits $W_B = T_1 + T_2 + \cdots + T_B$. Thus, on average, the i^{th} one waits $E(W_i) = E(T_1 + \cdots + T_i) = iE(T) = i/\lambda$, and so the sum of all B sojourn times has average

$$E\left(\sum_{i=1}^{B} W_i\right) = \frac{1}{\lambda}\sum_{i=1}^{B} i = \frac{1}{\lambda}\frac{B(B+1)}{2}.$$

Consequently, diving by B to obtain w:

$$w = E\left(\frac{1}{B}\sum_{i=1}^{B} W_i\right) = \frac{B+1}{2\lambda}.$$

Finally,

$$l = \overline{\lambda}w = \left(\frac{B}{\frac{B}{\lambda} + \frac{1}{\gamma}}\right)\frac{B+1}{2\lambda} = \frac{1}{B + \frac{\lambda}{\gamma}}\left((B)(B+1)/2\right).$$

(e) Suppose there are some costs incorporated: Each delivery cost of restocking back to B headphones is $\$K$, while there is also an inventory holding cost of $\$ci$ per unit time that $X(t) = i \geq 1$ items are in the inventory. Letting $C(t)$ denote the total cumulative cost incurred up to time t; obtain an expression for the long-run cost rate $\lim_{t\to\infty} C(t)/t$ using renewal reward.

SOLUTION:

$X = t_B + L$; $E(X) = B/\lambda + 1/\gamma$.

Letting $\{T_n\}$ denote iid exponential demand interarrival times,

$R = (cBT_1 + c(B-1)T_2 + \cdots + cT_B) + K$, and hence

$E(R) = cE(T)(1 + 2 + \cdots + B) + K = c/\lambda((B(B+1))/2 + K$.

$$g(B) = \frac{E(R)}{E(X)} = \frac{c/\lambda((B(B+1))/2 + K}{B/\lambda + 1/\gamma}.$$

(f) Using $c = \lambda = \gamma = 1$ and $K = 50$, find the optimal value of B (the one that minimizes cost).
SOLUTION:

$$g(B) = \frac{((B(B+1))/2 + K}{B+1} = \frac{B}{2} + \frac{K}{B+1}.$$

Setting $g'(B) = 0$ yields $(B+1)^2 = 2K$, with solution $B = \sqrt{2K} - 1 = \sqrt{100} - 1 = 9$. ($g''(B) > 0$ hence indeed we have a minimum.)

7. Consider a renewal process $\{t_n\}$ with iid interarrival times $\{X_n\}$ distributed as having probability density function

$$f(x) = 2x, \ x \in (0,1).$$

Imagine that they are the interarrival times of buses going downtown from the West 116th Street stop, here in NYC. You randomly way out in the future go to the stop to catch the next bus. Time is in hours.

(a) On average, what is your waiting time?
SOLUTION:

$$\frac{E(X^2)}{2E(X)} = \frac{3}{8},$$

where we compute

$$E(X) = \int_0^1 xf(x) = \int_0^1 2x^2 dx = \frac{2}{3}$$
$$E(X^2) = \int_0^1 x^2 f(x) = \int_0^1 2x^3 dx = \frac{1}{2}$$

(b) On average, what is the length of the interarrival time you landed in?
SOLUTION: (Average spread.)

$$\frac{E(X^2)}{E(X)} = \frac{3}{4},$$

twice the answer from (a).

(c) What is the probability that you must wait longer that 15 minutes (1/4 hour)?
SOLUTION:
We want (for $x = 1/4$) the tail of the equilibrium distribution

$$\overline{F}_e(x) = \lambda \int_x^\infty P(X > y)dy.$$

$\lambda = 1/E(X) = 3/2$ and for $y \in (0,1)$,

$$P(X > y) = \int_y^1 2xdx = 1 - y^2;$$

$$\overline{F}_e(x) = (3/2)\int_x^1 (1 - y^2)dy = \frac{2 - 3x + x^3}{2},$$

$$\overline{F}_e(1/4) = \frac{81}{128}.$$

8. *Train dispatching problem; different model*: Passengers arrive to a train platform according to a *Poisson process* at rate μ. A train departs every T time units, taking all passengers who arrived during the T time units. ($T > 0$ is a constant.) Suppose further that the train company incurs a cost at the constant rate of $\$nc$ per unit time whenever exactly n passengers are waiting, and also incurs a fixed cost of $\$K$ each time a train departs. This process continues over and over. Our objective in what follows is to compute (using Renewal Reward) the long-run cost rate for the train company. Observe that the cycle lengths are deterministic of length T; $X_n = T,\ n \geq 0$.

   (a) On average, how many passengers get on a train?
   SOLUTION:
   Letting $\{N(t)\}$ denote the Poisson counting process, we want $E(N(T)) = \mu T$.

   (b) On average, what is the waiting time of a passenger? (Hint: Condition on how many Poisson arrivals occur by time T; recall the use of order statistics, etc.)
   SOLUTION: $T/2$ as argued as follows:
   Given that $N(T) = n \geq 1$, the n arrival times are the order statistics $V_{(1)} < V_{(2)} < \cdots < V_{(n)}$ of n iid uniform rvs over $(0, T)$ denoted by $V_1, \ldots, V_n$. The waiting time of the i^{th} passenger is $W_i = t - V_{(i)}$, and we note that

$$A = \frac{1}{n}\sum_{i=1}^{n}(T - V_{(i)}) = \frac{1}{n}\sum_{i=1}^{n}(T - V_i),$$

   hence the expected values are the same; $E(A) = E(T - V) = T - E(V) = T/2$. This is true for any $n \geq 1$ hence is the answer in question.

   (c) What is the expected waiting cost per cycle *per passenger*? What is thus $E(R)$?
   SOLUTION: From (b) above, we know that the average waiting time of each passenger is $T/2$, hence the expected waiting cost per passenger is $cT/2$. There are $N(T)$ passengers total and hence $E(N(T) = \mu T$ passengers on average, and hence over a cycle the total expected waiting cost over all customers is $\mu T(cT/2) = \mu cT^2/2$. Finally adding in the cost K per cycle we get
   $E(R) = \mu cT^2/2 + K$.

   (d) Now compute $E(R)/E(X) =$ the long-run cost rate for the train company.
   SOLUTION:

$$g(T) = E(R)/E(X) = \frac{\mu cT^2/2 + K}{T} = \mu cT/2 + \frac{K}{T}.$$

   (e) Find the optimal value of T; the one that minimizes cost.
   SOLUTION: Letting $g(T) = c\mu T/2 + K/T$, we must solve $g'(T) = 0$: $g'(T) = c\mu/2 - K/T^2 = 0$, $T = \sqrt{2K/\mu c}$. It is a minimum since $g''(T) > 0$.