IEOR 4106, HMWK 7, Professor Sigman

1. For generations, a family (and their descendants) keep a Galapagos tortoise in their yard as a pet. As soon as it dies, they immediately get a new just born one. Suppose that the average lifetime of such a tortoise is 250 years, and the *standard deviation* is $\sigma = \sqrt{\sigma^2} = 30$. Suppose (way out in the infinite future) you visit the family. What is the expected *total lifetime* of the tortoise you find (e.g., sum of age plus remaining lifetime)? Is the inspection paradox holding here?

2. *Printer with disasters*: Jobs arrive to a computer printer according to a Poisson process at rate λ. Jobs are printed one at a time requiring iid printing times that are exponentially distributed with rate μ. Jobs wait in a FIFO queue before entering service.

   Additionally, independently, disasters occur according to a Poisson process at rate γ. Whenever a disaster occurs all jobs are removed and lost (both any in line and the one in service). If the printer has no jobs, then the disaster has no effect (e.g., the printer instantly resets, waiting for new arrivals). Let $X(t)$ denote the number of jobs at the printer at time t.

   (a) $\{X(t)\}$ is not a Birth and Death process (why?). But it is an irreducible CTMC. Draw the rate diagram. We will show below that the chain is always positive recurrent.

   (b) Set up the balance equations, with the first one as

   $$\lambda P_0 = \mu P_1 + \gamma(1 - P_0).$$

   Explain why this indeed is the first equation: "Rate out of state 0 equals rate into state 0". (You are not expected to solve these equations.......very difficult.....)

   (c) Explain why $E(T_{0,0}) \leq 1/\gamma < \infty$, hence the chain is always positive recurrent.

3. Consider the M/M/1 queue (arrival rate λ service time rate μ) with impatient customers: Each customer independently will get impatient after an amount of time that is exponentially distributed at rate γ while waiting in line (queue) and leave before ever entering service, and without ever returning. A customer who does enter service completes service (e.g., customers are only impatient while waiting in the line, not when in service.)

   (a) Set up the birth and death balance equations but do not try to solve in general.

   (b) Show that in the special case when $\gamma = \mu$, you can solve the birth and death balance equations that you set up in (a); solve them. What famous other CTMC model has these same birth and death balance equations?

4. Consider c (identical) ATM machines working in parallel with one FIFO line (queue). Customers arrive at an existing rate $\lambda < \infty$, and service times $\{S_n\}$ have an existing average of $1/\mu < \infty$. (Assume that $\lambda < c\mu$.) Letting the "system" be just the set of c servers, give an expression for the long-run average number of busy servers: If $Y(t) =$ the number of busy servers at time t, we want

   $$\lim_{t\to\infty} \frac{1}{t} \int_0^t Y(s)ds.$$

5. Consider a renewal process with iid interarrival times $\{X_n\}$ with finite (and non-zero) moments $E(X), E(X^2), E(X^3)$. Let $A(t)$ denote the forward recurrence time, $A(t) = t_{N(t)+1} - t,\ t \geq 0$.

(a) Graph $\{A^2(t) : t \geq 0\}$.

(b) Compute (wp1):

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t A^2(s)ds.$$

6. Recall the Inventory Model that you previously modeled as a CTMC in HMWK 6 (Problem 3): A retailer sells headphones one at a time according to demand which forms a Poisson process at rate λ: At Poisson arrival time t_n (n^{th} demand request), the inventory drops by 1 if the inventory is non-empty. If the inventory is empty at a request time, then nothing happens, that demand request is "lost". The amount in inventory starts off as $B \geq 2$. As soon as the Inventory drops down to 0, it will be re-stocked up to B after an exponential amount of time L (lead time) at rate γ, independent of the past. Again: during those L time units, all demand is lost. Let $X(t)$ denote the inventory level at time t. The state space is thus $\{0, 1, \ldots, B\}$.

Here we will use the Renewal Reward Theorem to re-derive the limiting probabilities $\{P_i : 0 \leq i \leq B\}$ and even allow for a general distribution of lead time L (does not have to be exponentially distributed.)

(a) With $X(0) = B$, let $s_1 = $ time until the inventory returns back to level B for the first time, let $s_2 = $ the time until the inventory returns back to level B for the second time, and in general, let $s_n = $ the time until the inventory returns back to level B for the nth time, $n \geq 1$. Note that $s_1 = t_B + L$. Argue that $\{s_n : n \geq 1\}$ forms a renewal point process, that is, letting the cycle lengths be denoted by $X_n = s_n - s_{n-1}$, $n \geq 1$ (with $s_0 \stackrel{def}{=} 0$), they are iid distributed as $X = X_1 = s_1 = t_B + L$.

(b) Let

$$P_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) = i\}ds,$$

denote the long-run proportion of time that the inventory is at level i, $0 \leq i \leq B$. (These were computed by solving the balance equations previously; we will compute differently now.) Letting for a given fixed i

$$R = R_1 = \int_0^{X_1} I\{X(s) = i\}ds,$$

(the total amount of time spent in state i during the first cycle) denote the "reward over the first cycle", argue that $E(R) = 1/\lambda$, $1 \leq i \leq B$ and $E(R) = 1/\gamma$, $i = 0$. (The point is that we can imagine that rewards are collected continuously at rate 1 whenever $X(t) = i$ and at rate 0 otherwise.) More generally, argue that for $j \geq 1$

$$R_j = \int_{s_{j-1}}^{s_j} I\{X(s) = i\}ds,$$

are iid rewards over the jth cycle, and that $\{(X_j, R_j)\}$ are iid.

(c) Letting, for a given fixed i

$$R(t) = \int_0^t I\{X(s) = i\}ds,$$

denote the total reward collected up until time t, derive an explicit expression for each P_i , using renewal reward, $P_i = \lim_{t\to\infty} R(t)/t = E(R)/E(X)$. (You should get exactly the same answer as when you solved the balance equations in Problem 1 of HMWK 8.)

(d) Suppose that we change the model to allow the iid lead times $L > 0$ to have a general distribution $G(x) = P(L \leq x)$, $x \geq 0$, with $E(L) = 1/\gamma < \infty$. In other words we do not assume they are exponentially distributed anymore. Explain why (in general now) $\{X(t) : t \geq 0\}$ is no longer a CTMC, but that the $\{P_j\}$ defined above still exist and are exactly the same value as when the L are exponentially distributed.

(e) Suppose there are some costs incorporated: Each delivery cost of restocking back to B headphones is $\$K$, while there is also an inventory holding cost of $\$ci$ per unit time that $X(t) = i \geq 1$ items are in the inventory. Letting $C(t)$ denote the total cumulative cost incurred up to time t; obtain an expression for the long-run cost rate $\lim_{t\to\infty} C(t)/t$ using renewal reward.

(f) Using $c = \lambda = \gamma = 1$ and $K = 50$, find the optimal value of B (the one that minimizes cost).

7. Consider a renewal process $\{t_n\}$ with iid interarrival times $\{X_n\}$ distributed as having probability density function
$$f(x) = 2x, \ x \in (0, 1).$$

Imagine that they are the interarrival times of buses going downtown from the West 116th Street stop, here in NYC. You randomly way out in the future go to the stop to catch the next bus. Time is in hours.

(a) On average, what is your waiting time?

(b) On average, what is the length of the interarrival time you landed in?

(c) What is the probability that you must wait longer that 15 minutes (1/4 hour)?

8. *Train dispatching problem; different model*: Passengers arrive to a train platform according to a *Poisson process* at rate μ. A train departs every T time units, taking all passengers who arrived during the T time units. ($T > 0$ is a constant.) Suppose further that the train company incurs a cost at the constant rate of $\$nc$ per unit time whenever exactly n passengers are waiting, and also incurs a fixed cost of $\$K$ each time a train departs. This process continues over and over. Our objective in what follows is to compute (using Renewal Reward) the long-run cost rate for the train company. Observe that the cycle lengths are deterministic of length T; $X_n = T$, $n \geq 0$.

(a) On average, how many passengers get on a train?

(b) On average, what is the waiting time of a passenger? (Hint: Condition on how many Poisson arrivals occur by time T; recall the use of order statistics, etc.)

(c) What is the expected waiting cost per cycle *per passenger*? What is thus $E(R)$?

(d) Now compute $E(R)/E(X) =$ the long-run cost rate for the train company.

(e) Find the optimal value of T; the one that minimizes cost.