IEOR 4106, HMWK 8, Professor Sigman

1. The price of a commodity moves according to a BM, $X(t) = 1 + 2B(t) + 5t,\ t \geq 0$.

   (a) What is the mean and variance of the price at time $t = 4$?
   SOLUTION:
   $E(X(4)) = E(1 + 2B(4) + 20) = 21 + 2E(B(4)) = 21 + 0 = 21.$
   $Var(X(4)) = Var(1 + 2B(4) + 20) = Var(2(B(4)) = 4Var(B(4)) = 4^2 = 16.$

   (b) What is the probability that at time $t = 4$, the price is > 14?
   SOLUTION:
   $X(4) > 14$ if and only if $21 + 2B(4) > 14$. $2B(4)$ is $N(0, 16)$ hence can be re-written as $4Z$, where $Z \sim \mathcal{N}(0, 1)$. Thus we want $P(21 + 4Z > 14) = P(Z > -(7/4)) = P(Z > -1.75) = P(Z \leq 1.75) = \Phi(1.75) = 0.96$.

   (c) Given that the price is 5.5 at time $t = 6$, what is the probability that the price is > 8 at time $t = 7$?
   SOLUTION:
   $X(7) = X(6) + (X(7) - X(6)) = 5.5 + (5 + 2(B(7) - B(6)) = 10.5 + 2(B(7) - B(6))$. By stationary and independent increments, this has the same distribution as $10.5 + 2B(1)$ which has the same distribution as $10.5 + 2Z$, where $Z \sim \mathcal{N}(0, 1)$.
   So we want $P(10.5 + 2Z > 8) = P(Z > -1.25) = P(Z \leq 1.25) = \Phi(1.25) = 0.89$.

   (d) What is the probability the price goes up to 4 before down to 1/4?
   SOLUTION:
   $X(t) = 1 + 2B(t) + 5t$ hits 4 before 1/4 if and only if $2B(t) + 5t$ hits $a = 3$ before $-b = -3/4$. With $\mu = 5$ and $\sigma = 2$ we get $2\mu/\sigma^2 = (2 \times 5)/4 = 2.5$.

$$p_a = \frac{1 - e^{(2\mu/\sigma^2)b}}{e^{(-2\mu/\sigma^2)a} - e^{(2\mu/\sigma^2)b}} = \frac{1 - e^{(15/8)}}{e^{((-15/2))} - e^{(15/8)}} = 0.8467.$$

2. Let Z denote a standard unit normal random variable. Let $X(t) = \sqrt{t}Z,\ t \geq 0$. Carefully explain your answers below.

   (a) Does $\{X(t) : t \geq 0\}$ have continuous sample paths?
   SOLUTION:
   Yes. For each realization $Z = z$, the function $\sqrt{t}z,\ t \geq 0$ is continuous in t.

   (b) Does $X(t_2) - X(t_1)$ have a normal distribution for all $0 \leq t_1 < t_2$?
   SOLUTION:
   Yes: $X(t_2) - X(t_1) = \sqrt{t_2}Z - \sqrt{t_1}Z = (\sqrt{t_2} - \sqrt{t_1})Z$; hence is $N(0, (\sqrt{t_2} - \sqrt{t_1})^2)$.

   (c) Is $\{X(t) : t \geq 0\}$ a standard BM?
   SOLUTION: No. There are many ways to prove this. For example, in (b), although the increment has a normal distribution, the variance is wrong. For BM, the variance must be $t_2 - t_1$, not $(\sqrt{t_2} - \sqrt{t_1})^2$. (This then even shows that $\{X(t)\}$ does not have stationary increments: both $(t_1, t_2) = (1, 2)$, and $(t_1, t_2) = (2, 3)$ yield increments of length 1, but the variance is now $(\sqrt{t_2} - \sqrt{t_1})^2$ for each, and is not the same for the two as would be required for stationary increments.)
   The most fundamental/obvious thing that is wrong is *independent increments*: every increment of $X(t)$ has the same random variable Z in it; hence all increments are dependent on Z.
   Although for each *fixed* $t > 0$, $X(t)$ and $B(t)$ have the same ($N(0, t)$) distribution; as stochastic processes, they are completely different.

3. (a) Let $\{B(t) : t \geq 0\}$ denote standard BM. Show that $\{-B(t) : t \geq 0\}$ is also a standard BM, by arguing that it satisfies the Properties of Definition 1.1 in your Lecture Notes on Brownian Motion. (e.g., it has continuous sample paths, stationary and independent increments, etc.)

SOLUTION:

Letting $H(t) = -B(t),\ t \geq 0$, $\{H(t) : t \geq 0\}$ still has continuous sample paths and $H(0) = 0$ and the increments are both stationary and independent; and $H(t) - H(s)$ is $N(0, t - s)$, $0 \leq s < t$.

(b) *Continuation:* If $X_1(t) = \sigma_1 B_1(t) + \mu_1 t$ and $X_2(t) = \sigma_2 B_2(t) + \mu_2 t$ are independent BM's, then argue that $X(t) \stackrel{\text{def}}{=} X_1(t) - X_2(t)$ is also a BM with $\sigma =?$ and $\mu =?$

SOLUTION: As in (a), we just check the requirements: Continuous sample paths still hold for $X(t)$ (the addition or subtraction of two continuous functions is always continuous). Both stationary and independent increments holds since it does for each of $X_1(t)$ and $X_2(t)$, and the two are assumed independent. That in fact is enough to prove that it is a BM (and hence has normally distributed increments), but we can see directly that $X(t)$ has a normal distribution for each t since the sum/difference of independent normal random variables is normal.

$\mu = E(X(1)) = E(X_1(1)) - E(X_2(1)) = \mu_1 - \mu_2$,

$\sigma^2 = Var(X(1)) = Var(X_1(1)) + Var(X_2(1)) = \sigma_1^2 + \sigma_2^2$. (Thus for $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$, and $\mu = \mu_1 - \mu_2$, we can represent $X(t) = \sigma B(t) + \mu t, t \geq 0$ for a standard BM $\{B(t)\}$.

(c) Let $\{B(t) : t \geq 0\}$ denote standard BM. What is the probability that $\{B(t)\}$ hits -6 before hitting 8?

SOLUTION: $a = 8,\ b = 6$, we want $p(b) = a/(a + b) = 4/7$.

(d) *Continuation:* Suppose that $B(2) = 4$. What is the probability that $\{B(2+t) : t \geq 0\}$ hits -6 before hitting 8?

SOLUTION: Now this is equivalent to assuming that $B(0) = 0$ and we want the probability that $\{B(t)\}$ hits $-10 = -6 - 4$ before $4 = 8 - 4$ (e.g., going down by 10 before going up by 4.) $a = 4,\ b = 10$, we want $p(b) = a/(a + b) = 2/7$.

(e) Suppose a stock price per share moves as $S(t) = e^{B(t)},\ t \geq 0$. What is the probability that $\{S(t)\}$ goes up to 3 before down to 1/3?

SOLUTION:

Taking natural logarithms, this is equivalent to $B(t)$ hits $\ln(3)$ before $\ln(1/3) = -\ln(3)$; $a = b = \ln(3)$; $p(a) = 1/2$.

(f) If $X_1(t) = 2B_1(t)+5t$ with $X_1(0) = 4$ is a BM, and independently $X_2(t) = 3B_2(t)+3t$ with $X_2(0) = 0$ is another BM, compute the probability that in the future the two processes meet (e.g, that eventually $X_1(t) = X_2(t)$ for some t.) HINT: Use (b) above.

SOLUTION: $X_1(t) = 4 + 2B_1(t) + 5t,\ t \geq 0$, and $X_2(t) = 3B_2(t) + 3t,\ t \geq 0$. $X_1(t) = X_2(t)$ for some t if and only if $(3B_2(t) + 3t) - (2B_1(t) + 5t) = 4$, for some t. From (b) above we have that $X(t) = (3B_2(t) + 3t) - (2B_1(t) + 5t)$ is a BM with $\mu = 3 - 5 = -2$ and $\sigma^2 = 3^2 + 2^2 = 13$, hence can be represented by $X(t) = \sqrt{13}B(t) - 2t,\ t \geq 0$. Because of continuous sample paths, $X(t) = 4$ for some $t \geq 0$ if and only if

$$P(M \geq 4),$$

where

$$M = \max\{X(t) : t \geq 0\}.$$

M has an exponential distribution with rate $\alpha = 2|\mu|/\sigma^2 = 4/13$;

$$P(M \geq 4) = e^{-4(4/13)} = e^{-(16/13)} = 0.292.$$

4. With $n \geq 2$ fixed, let $X_1, \ldots, X_n$ be iid rvs distributed as $F(x) = P(X \leq x),\ x \geq 0$, and assume that $F^{-1}(y),\ y \in [0, 1]$, is known in closed form. Suppose that we want to generate a rv distributed as $M = \max\{X_1, X_2, \ldots, X_n\}$. We could simply generate $X_i = F^{-1}(U_i)$ for n iid uniforms and set $M = \max\{X_1, X_2, \ldots, X_n\}$. That would require n iid uniforms to generate one copy of M. But let us explore another method:

   (a) Give an algorithm for generating a rv $M = \max\{X_1, X_2, \ldots, X_n\}$ that uses only ONE uniform U.
   SOLUTION: Generate U and set $M = F^{-1}(U^{1/n})$ as derived as follows: We compute $F_M(x) = P(M \leq x)$ and then $F_M^{-1}(y)$ and use the inverse transform method: $M = F_M^{-1}(U)$.

   $M \leq x$ if and only if $X_i \leq x$ for all $1 \leq i \leq n$, hence by independence $F_M(x) = P(M \leq x) = P(X_1 \leq x) \cdots P(X_n \leq x) = F^n(x)$. Setting $y = F^n(x)$ and solving for x in terms of y:
   $y^{1/n} = F(x)$ and thus via taking F^{-1} on both sides yields $F_M^{-1}(y) = x = F^{-1}(y^{1/n})$. Thus generate U and set $M = F^{-1}(U^{1/n})$.

   (b) Give the algorithm in the special case when $F(x) = 1 - e^{-\lambda x},\ x \geq 0$, the exponential distribution at rate λ.
   SOLUTION: In this case $F^{-1}(y) = -\frac{1}{\lambda}\ln(1 - y)$. Thus

   $$M = F_M^{-1}(U) = -\frac{1}{\lambda}\ln(1 - U^{1/n}).$$

   NOTE: we can't replace $1 - y$ by y as we do for the exponential distribution itself; whereas $1 - U$ has the same distribution as U, it is not true that $1 - U^{1/n}$ has the same distribution as $U^{1/n}$.

5. Consider two independent geometric Brownian motions:

   $$S_1(t) = 3e^{4B_1(t)+2t},\ S_2(t) = 4e^{2B_2(t)+5t},\ t \geq 0.$$

   (a) What is the probability that $S_1(2) \geq 5$?
   SOLUTION: Taking natural logarithms, and letting Z denote a standard unit normal, we want

   $$P(\ln(3) + 4B_1(2) + 4 \geq \ln(5) = P(\ln(3) + 4\sqrt{2}Z + 4 \geq \ln(5)).$$

   This becomes

   $$P(Z \geq \frac{\ln(5/3) - 4}{4\sqrt{2}}) = P(Z \geq -0.6168) = P(Z \leq 0.6168) = 0.7313$$

   (b) What is the probability that $S_1(t)$ will ever have its price $\geq 2S_2(t)$ at some time in the future?
   SOLUTION:

We need to compute

$$P(\frac{S_1(t)}{S_2(t)} \geq 2, \text{some } t).$$

To this end:

$$\frac{S_1(t)}{S_2(t)} = \frac{3}{4} e^{X_1(t) - X_2(t)},$$

where (by Problem 3(b) above) $X_1(t) - X_2(t) = 4B_1(t) + 2t - (2B_2(t) + 5t)$ is a BM and has the same distribution as $X(t) = \sqrt{20}B(t) - 3t$.
Taking natural logarithms, we need to compute

$$P(M \geq \ln(8/3)),$$

where

$$M = \max_{t \geq 0} \sqrt{20}B(t) - 3t.$$

For $\alpha = \frac{2|\mu|}{\sigma^2} = 3/10$, we want $P(M > a) = e^{-\alpha a}$ for $a = \ln(8/3)$; 0.9066.