IEOR 4106, HMWK 8, Professor Sigman

1. The price of a commodity moves according to a BM, $X(t) = 1 + 2B(t) + 5t,\ t \geq 0$.
   (a) What is the mean and variance of the price at time $t = 4$?
   (b) What is the probability that at time $t = 4$, the price is > 14?
   (c) Given that the price is 5.5 at time $t = 6$, what is the probability that the price is > 8 at time $t = 7$?
   (d) What is the probability the price goes up to 4 before down to 1/4?

2. Let Z denote a standard unit normal random variable. Let $X(t) = \sqrt{t}Z,\ t \geq 0$. Carefully explain your answers below.
   (a) Does $\{X(t) : t \geq 0\}$ have continuous sample paths?
   (b) Does $X(t_2) - X(t_1)$ have a normal distribution for all $0 \leq t_1 < t_2$?
   (c) Is $\{X(t) : t \geq 0\}$ a standard BM?

3. (a) Let $\{B(t) : t \geq 0\}$ denote standard BM. Show that $\{-B(t) : t \geq 0\}$ is also a standard BM, by arguing that it satisfies the Properties of Definition 1.1 in your Lecture Notes on Brownian Motion. (e.g., it has continuous sample paths, stationary and independent increments, etc.)
   (b) *Continuation:* If $X_1(t) = \sigma_1 B_1(t) + \mu_1 t$ and $X_2(t) = \sigma_2 B_2(t) + \mu_2 t$ are independent BM's, then argue that $X(t) \stackrel{\text{def}}{=} X_1(t) - X_2(t)$ is also a BM with $\sigma =$? and $\mu =$?
   (c) Let $\{B(t) : t \geq 0\}$ denote standard BM. What is the probability that $\{B(t)\}$ hits -6 before hitting 8?
   (d) *Continuation:* Suppose that $B(2) = 4$. What is the probability that $\{B(2+t) : t \geq 0\}$ hits -6 before hitting 8?
   (e) Suppose a stock price per share moves as $S(t) = e^{B(t)},\ t \geq 0$. What is the probability that $\{S(t)\}$ goes up to 3 before down to 1/3?
   (f) If $X_1(t) = 2B_1(t) + 5t$ with $X_1(0) = 4$ is a BM, and independently $X_2(t) = 3B_2(t) + 3t$ with $X_2(0) = 0$ is another BM, compute the probability that in the future the two processes meet (e.g, that eventually $X_1(t) = X_2(t)$ for some t.) HINT: Use (b) above.

4. With $n \geq 2$ fixed, let $X_1, \ldots, X_n$ be iid rvs distributed as $F(x) = P(X \leq x),\ x \geq 0$, and assume that $F^{-1}(y),\ y \in [0, 1]$, is known in closed form. Suppose that we want to generate a rv distributed as $M = \max\{X_1, X_2, \ldots, X_n\}$. We could simply generate $X_i = F^{-1}(U_i)$ for n iid uniforms and set $M = \max\{X_1, X_2, \ldots, X_n\}$. That would require n iid uniforms to generate one copy of M. But let us explore another method:
   (a) Give an algorithm for generating a rv $M = \max\{X_1, X_2, \ldots, X_n\}$ that uses only ONE uniform U.
   (b) Give the algorithm in the special case when $F(x) = 1 - e^{-\lambda x},\ x \geq 0$, the exponential distribution at rate λ

5. Consider two independent geometric Brownian motions:
$$S_1(t) = 3e^{4B_1(t)+2t},\ S_2(t) = 4e^{2B_2(t)+5t},\ t \geq 0.$$
   (a) What is the probability that $S_1(2) \geq 5$?
   (b) What is the probability that $S_1(t)$ will ever have its price $\geq 2S_2(t)$ at some time in the future?