The Binomial Lattice Model for Stocks: Introduction to Option Pricing

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Pricing Options: Matching Portfolio Method

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Pricing Options: Matching Portfolio Method

Black-Scholes-Merton Option-Pricing Formula (for European Call Options)

The Model

Definition

The Binomial Lattice Model (BLM) is a Markovian stochastic process $\{S_n : n \geq 0\}$ defined recursively via

$$S_{n+1} = S_n Y_{n+1}, \; n \geq 0,$$

where $S_0 > 0$ is the initial value, and for fixed probability $0 < p < 1$, the random variables (rvs) $\{Y_n : \; n \geq 1\}$ form an independent and identically distributed (iid) sequence distributed as the two-point “up" (u), “down" (d) distribution:

$$P(Y = u) = p, \; P(Y = d) = 1 - p,$$

with $0 < d < 1 + r < u$, where $r > 0$ is the risk-free interest rate.

The Model 2

In our stock application here: S_0 denotes the initial price per share at time $t = 0$; S_n denotes the price at the end of the n^{th} day. Each day, independent of the past, the stock either goes UP with probability p, or it goes DOWN with probability $1 - p$. For example,

$$
\begin{aligned}
S_1 &= uS_0, \text{ with probability } p \\
&= dS_0, \text{ with probability } 1 - p.
\end{aligned}
$$

Similarly, one day later at time $t = 2$:

$$
\begin{aligned}
S_2 &= u^2 S_0, \text{ with probability } p^2 \\
&= duS_0, \text{ with probability } (1 - p)p \\
&= udS_0, \text{ with probability } p(1 - p) \\
&= d^2 S_0, \text{ with probability } (1 - p)^2.
\end{aligned}
$$

Interest rate r

The idea is that if the stock goes ‘up’, we mean that is does better than the bank account interest rate, whereas if it goes ‘down’, we mean it does worse than the bank account interest rate:
$uS_0 > (1 + r)S_0$ and $dS_0 < (1 + r)S_0$
This all makes economic sense.

Interest rate r

We assume that one can't make a fortune from nothing (arbitrage). Let $r > 0$ denote the interest rate that we all have access to if we placed our money in a savings account. If we started with an initial amount x_0 placed in the account, then one day later it would be worth $x_1 = (1 + r)x_0$ and in general, n days later it would be worth $x_n = (1 + r)^n x_0$. Thus the *present value* of x_n (at time n) is $x_0 = x_n/(1 + r)^n$.
From these basic principles, it must hold that

$$0 < d < 1 + r < u$$

because otherwise there would be arbitrage.

The Model 2

In general, the *Binomial Distribution* governs the movement of the prices over time: At any time $t = n$: For any $0 \le i \le n$,

$$
\begin{aligned}
P(S_n = u^i d^{n-i} S_0) &= \text{The prob that during the first } n \text{ days} \\
&\quad\ \text{the stock went up } i \text{ times (thus down } n-i \text{ times)} \\
&= \text{The "probability of } i \text{ successes out of } n \text{ trials"} \\
&= \binom{n}{i} p^i (1-p)^{n-i}.
\end{aligned}
$$

Also, the space of values that this process can take is given by the *lattice of points*:

$$\{S_0 u^i d^j : \ i \ge 0, \ j \ge 0\}.$$

That is why this model is called the *Binomial Lattice Model.*.

The Model 3

The recursion can be expanded yielding:

$$S_n = S_0 Y_1 x \cdots x Y_n, \; n \geq 0.$$

This makes it easy to do simple computations such as expected values: Noting that the expected value of a Y random variable is given by

$$E(Y) = pu + (1 - p)d,$$

we conclude from independence that the expected price of the stock at the end of day n is

$$E(S_n) = S_0 E(Y)^n = S_0[pu + (1 - p)d]^n, \; n \geq 1.$$

Simulating a Y (up/down) random variable

If we want to simulate a Y rv such that $P(Y = u) = p,\ P(Y = d) = 1 - p$, we can do so with the following simple algorithm:

(i) Enter p, u, d

(ii) Generate a U

(iii) Set

$$Y = \begin{cases} u & \text{if } U \le p\ , \\ d & \text{if } U > p. \end{cases}$$

Simulating the BLM

We then can simulate the BLM by using the recursion $S_{i+1} = S_i Y_{i+1}$, $0 \leq i \leq n-1$:

(i) Enter p, u, d, and S_0, and n. Set $i = 1$

(ii) Generate U

(iii) Set

$$Y = \begin{cases} u & \text{if } U \leq p \,, \\ d & \text{if } U > p. \end{cases}$$

(iv) Set $S_i = S_{i-1} Y$. If $i < n$, then reset $i = i + 1$ and go back to (ii); otherwise stop.

1. Output $S_0, S_1, \ldots, S_n$.

The Model 4

In real situations, $E(Y) >> 1 + r$, so that

$$E(S_n) >> S_0(1 + r)^n$$

: *On* **AVERAGE**, *the price of the stock goes up by much more than just putting your money in the bank, compounded daily at fixed interest rate r. You expect to make a lot of profit over time from your investment of S_0.*
If you initially buy α shares of the stock, at a cost of αS_0, you will have, on average, $\alpha S_0 E(Y)^n >> \alpha S_0(1 + r)^n$ amount of money after n days. This is why people invest in stocks. **But of course, unlike a fixed interest rate r, buying stock has significant *risk* associated with it, because of the randomness involved. The stock might drop in price causing you to lose a fortune.**

Options of the stock 1

Definition

A European Call Option with expiration date $t = T$, and strike price K gives you a (random) payoff C_T at time T of the amount

$$\text{Payoff at time } T = C_T = (S_T - K)^+,$$

where $x^+ = \max\{0, x\}$ is the positive part of x.

The meaning: If you buy this option at time $t = 0$, then it gives you the right (the "option") of buying 1 share of the stock at time T at price K. If $K < S_T$ (the market price), then you will exercise the option (buy at cheaper price K) and immediately sell it at the higher market price to make the profit $S_T - K > 0$. Otherwise you will not exercise the option and will make no money (payoff$= 0$.)

Options of the stock 2

Whereas we know the stock price at time $t = 0$; it is simply the market price S_0, *we do not know (yet) what a fair price C_0 should be for this option.*

Since $C_T \leq S_T$, it must hold that $C_0 \leq S_0$: **The price of the option should be cheaper than the price of the stock since its payoff is less.**

But what should the price be exactly? How can we derive it?

Options of the stock 3

We consider first, the case when $T = 1$; $C_T = C_1 = (S_1 - K)^+$. Then, if the stock goes up,

$$C_1 = C_u = (uS_0 - K)^+,$$

and if the stock goes down, then

$$C_1 = C_d = (dS_0 - K)^+.$$

Note that

$$E(C_1) = pC_u + (1 - p)C_d,$$

is the expected payoff.

Matching Portfolio Method 1

Consider as an alternative investment, a *portfolio* (α, β) of α shares of stock and placing β amount of money in the bank at interest rate r, all at time $t = 0$ at a cost (price) of exactly

$$\text{Price of the portfolio} = \alpha S_0 + \beta.$$

Then, at time $T = 1$, the payoff $C_1(P)$ of this portfolio is the (random) amount

$$\text{Payoff of portfolio} = C_1(P) = \alpha S_1 + \beta(1 + r).$$

Then, if the stock goes up,

$$C_1(P) = C_u(P) = \alpha u S_0 + \beta(1 + r),$$

and if the stock goes down, then

$$C_1(P) = C_d(P) = \alpha d S_0 + \beta(1 + r).$$

Matching Portfolio Method 2

We now will choose the values of α and β so that the two payoffs $C_1(P)$ and C_1 are the same, that is they *match*. **Choose** $\alpha = \alpha^*$ **and** $\beta = \beta^*$ **so that**

$$C_1(P) = C_1.$$

If they have the same payoff, then they must have the same price:

$$C_0 = \alpha^* S_0 + \beta^*.$$

But this happens if and only if the two payoff outcomes (up, down) match:

$$\begin{aligned} C_u(P) &= \alpha u S_0 + \beta(1+r) = C_u, \\ C_d(P) &= \alpha d S_0 + \beta(1+r) = C_d. \end{aligned}$$

Matching Portfolio Method 3

There is *always* a solution:

$$\alpha = \alpha^* = \frac{C_u - C_d}{S_0(u-d)} \qquad (1)$$

$$\beta = \beta^* = \frac{uC_d - dC_u}{(1+r)(u-d)}. \qquad (2)$$

Plugging this solution into $C_0 = \alpha^* S_0 + \beta^*$. yields

$$C_0 = \frac{C_u - C_d}{(u-d)} + \frac{uC_d - dC_u}{(1+r)(u-d)}.$$

Matching Portfolio Method 4

But we can simplify this algebraically (go check!) to obtain:

$$C_0 = \frac{1}{1+r}(p^* C_u + (1-p^*)C_d),$$

where

$$\begin{aligned} p^* &= \frac{1+r-d}{u-d} \\ 1-p^* &= \frac{u-(1+r)}{u-d}. \end{aligned}$$

Since $0 < d < 1 + r < u$ (by assumption), we see that $0 < p^* < 1$ is indeed a probability!

Matching Portfolio Method 5

C_0 is thus expressed elegantly as the *discounted expected payoff* of the option if $p = p^*$ for the underlying “up" probability p for the stock;

$$C_0 = \frac{1}{1+r} E^*(C_1), \tag{3}$$

where E^* denotes expected value when $p = p^*$ for the stock price. p^* is called the *risk-neutral* probability.

Matching Portfolio Method 5B

Note that our derivation would work for *any option* for which the payoff is at time $T = 1$ and for which we know the two payoff values $C_1 = C_u$ if the stock goes up, and $C_1 = C_d$ if the stock goes down. The European Call option is just one such an example. Summarizing: For any such option

$$C_0 = \frac{1}{1+r} E^*(C_1), \tag{4}$$

where E^* denotes expected value when $p = p^*$ for the stock price. In general $E(C_1) = pC_u + (1-p)C_d$, where p is the real up down probability; but when pricing options it is replaced by p^*.

Matching Portfolio Method 6

It is easily shown that p^* is the unique value of p so that $E(S_n) = S_0(1+r)^n$, that is, the unique value of p such that $E(Y) = pu + (1-p)d = 1 + r$. To see this, simply solve (for p) the equation $pu + (1-p)d = 1 + r$, and you get

$$p^* = \frac{1 + r - d}{u - d}.$$

p^* is the unique value of p that makes the stock price, on average, move exactly as if placing S_0 in the bank at interest rate r.
$E(S_n) = S_0(1+r)^n$

Matching Portfolio Method 7

When $T > 1$, the same result holds:

$$C_0 = \frac{1}{(1+r)^T} E^*(C_T). \tag{5}$$

$C_0 =$ *the discounted (over T time units) expected payoff* of the option if $p = p^*$. For example, when $T = 2$, there are 4 possible values for C_2: $C_{2,uu}, C_{2,ud}, C_{2,du}, C_{2,dd}$ corresponding to how the stock moved over the 2 time units ($u =$up, $d =$down). The corresponding (real) probabilities of the 4 outcomes is: $p^2, p(1-p), (1-p)p, (1-p)^2$, and so (in general, order matters for option payoffs):

$$E(C_2) = p^2 C_{2,uu} + p(1-p)C_{2,ud} + (1-p)pC_{2,du} + (1-p)^2 C_{2,dd}.$$

Matching Portfolio Method 8

The proof is quite clever: We illustrate with $T = 2$. Although we are not allowed to exercise the option at the earlier time $t = 1$, we could sell it at that time. Its worth/price would be the same as a $T = 1$ option price but with the stock having initial price S_1 instead of S_0. At time $t = 1$ we would know if $S_1 = uS_0$ or $S_1 = dS_0$. Let $C_{1,u}$ and $C_{1,d}$ denote the price at time $t = 1$; we will compute them using our $T = 1$ result.

Matching Portfolio Method 9

If at time $t = 1$, the stock went up ($S_1 = uS_0$) , then at time $T = 2$ (one unit of time later) we have the two possible prices of the stock; $S_2 = u^2 S_0, \; S_2 = duS_0$. So, using the $T = 1$ option pricing formula, we would obtain

$$C_{1,u} = \frac{1}{1+r}(p^* C_{2,uu} + (1 - p^*)C_{2,du}).$$

Similarly,

$$C_{1,d} = \frac{1}{1+r}(p^* C_{2,ud} + (1 - p^*)C_{2,dd}).$$

Matching Portfolio Method 9

But now, we can go back to time 0 to get C_0 by using the $T = 1$ formula yet again for an option that has initial price S_0 but payoff values $C_{1,u}$ and $C_{1,d}$:

$$C_0 = \frac{1}{1+r}(p^* C_{1,u} + (1 - p^*)C_{1,d}).$$

Expanding yields

$$C_0 = \frac{1}{(1+r)^2}((p^*)^2 C_{2,uu} + p^*(1-p^*)C_{2,ud} + (1-p^*)p^* C_{2,du} + (1-p^*)^2 C_{2,dd}),$$

which is exactly

$$\frac{1}{(1+r)^2} E^*(C_2).$$

Black-Scholes-Merton formula

If we apply this formula to the European call option, where $C_T = (S_T - K)^+$, (and order does *not* matter) then we obtain

Theorem (Black-Scholes-Merton)

$$
\begin{aligned}
C_0 &= \frac{1}{(1+r)^T} E^*(C_T) && (6) \\
&= \frac{1}{(1+r)^T} E^*(S_T - K)^+ && (7) \\
&= \frac{1}{(1+r)^T} \sum_{i=0}^{T} \binom{T}{i} (p^*)^i (1-p^*)^{T-i} (u^i d^{T-i} S_0 - K)^+. && (8)
\end{aligned}
$$

Black-Scholes-Merton formula

This explicitly gives the price C_0 for the European call option, which is why it is famous. In general, for other options, obtaining an explicit expression for C_0 is not possible, because we are not able to explicitly compute $E^*(C_T)$. The main reason is that for other options, order matters for the ups and downs during the T time units. For the European call, however, order does not matter: the payoff $C_T = (S_T - K)^+$ only depends (from the stock) on S_T and hence only on how many times the stock went up (and how many times it went down) during the T time units; e.g., “How many successes out of T Bernoulli trials".

Black-Scholes-Merton formula

For example, consider the *Asian* call option, with payoff

$$C_T = \left(\frac{1}{T} \sum_{i=1}^{T} S_i - K \right)^+ .$$

Here, the T values are summed up first and averaged before subtracting K. The sum depends on order, not just the number of ups and downs. For example, if $S_0 = 1$, and $T = 2$, then an up followed by a down yields $S_1 + S_2 = u + du$, while if a down follows an up we get $S_1 = d + du$., which is different since $d < u$. This is an example of a *path-dependent* option; the payoff depends on the whole path $S_0, S_1, \ldots, S_T$, not just S_T.

Monte Carlo Simulation

But we can always estimate expected values with great accuracy by using Monte Carlo simulation: Generate n (large) iid copies of C_T, denoted by $X_1, \dots, X_n$ (with $p = p^*$ in this case) and use

$$E^*(C_T) \approx \bar{X}(n) = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

This then gives our option price estimate as

$$C_0 \approx \frac{1}{(1+r)^T}\,\bar{X}(n).$$

Monte Carlo Simulation

The accuracy can be expressed by the use of *confidence intervals* because of the Strong Law of Large Numbers and the Central Limit Theorem.

$$\bar{X}(n) \pm z_{\alpha/2} \frac{s(n)}{\sqrt{n}},$$

yields a $100(1-\alpha)\%$ confidence interval, where (Z representing a standard unit normal r.v.) $z_{\alpha/2}$ is chosen so that $P(Z > z_{\alpha/2}) = \alpha/2$, and $s(n) = \sqrt{s^2(n)}$ is the sample standard deviation, where $s^2(n)$ denotes the sample variance for $X_1, \ldots, X_n$.

Monte Carlo Simulation

For example, when $\alpha = 0.05$ we a get a 95% confidence interval for $E^*(C_T)$:

$$\bar{X}(n) \pm 1.96 \frac{s(n)}{\sqrt{n}}.$$

We interpret this as that this interval contains/covers the true value $E^*(C_T)$ with probability 0.95.
The beauty of this is that we can choose huge values of n such as $10,000$ or larger (because we are simply simulating them) which thus ensures use of the Central Limit Theorem. This is different from when we use confidence intervals in statistics in which we must go out and collect the data, which might be very scarce, and hence only (say) $n = 30$ samples are available.