1 Notes on the Poisson Process

We present here the essentials of the Poisson point process with its many interesting properties. As preliminaries, we first define what a point process is, define the renewal point process and state and prove the Elementary Renewal Theorem.

1.1 Point Processes

Definition 1.1 *A simple point process $\psi = \{t_n : n \geq 1\}$ is a sequence of strictly increasing points*

$$0 < t_1 < t_2 < \cdots, \qquad (1)$$

with $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$. With $N(0) \stackrel{\text{def}}{=} 0$ we let $N(t)$ denote the number of points that fall in the interval $(0, t]$; $N(t) = \max\{n : t_n \leq t\}$. $\{N(t) : t \geq 0\}$ is called the counting process for ψ. If the t_n are random variables then ψ is called a random point process. We sometimes allow a point t_0 at the origin and define $t_0 \stackrel{\text{def}}{=} 0$. $X_n = t_n - t_{n-1},\ n \geq 1$, is called the n^{th} interarrival time.

We view t as time and view t_n as the n^{th} *arrival* time (although there are other kinds of applications in which the points t_n denote locations in space as opposed to time). The word *simple* refers to the fact that we are not allowing more than one arrival to ocurr at the same time (as is stated precisely in (1)). (Later we will allow more than one arrival at the same time, known as "batch arrivals" such as busloads.) In many applications there is a "system" to which customers are arriving over time (classroom, bank, hospital, supermarket, airport, etc.), and $\{t_n\}$ denotes the arrival times of these customers to the system. But $\{t_n\}$ could also represent the times at which phone calls are received by a given phone, the times at which jobs are sent to a printer in a computer network, the times at which a claim is made against an insurance company, the times at which one receives or sends email, the times at which an earthquake occurs, the times at which one sells or buys stock, the times at which a given web site receives hits, or the times at which subways arrive to a station. (Instead of time, we also can consider a *spacial* point process, meaning that the points t_n are located in $\mathbb{R}$ such as the locations of gas stations along a straight highway. But in the present notes we will primarily consider the points in time, and refer to $\{t_n\}$ as an *arrival process.*

Note that

$$t_n = X_1 + \cdots + X_n,\ n \geq 1,$$

the n^{th} arrival time is the sum of the first n interarrival times.

Also note that the event $\{N(t) = 0\}$ can be equivalently represented by the event $\{t_1 > t\}$, and more generally

$$\{N(t) = n\} = \{t_n \leq t, t_{n+1} > t\},\ n \geq 1.$$

In particular, for a random point process, $P(N(t)=0) = P(t_1 > t)$.

1.2 Renewal process

A random point process $\psi = \{t_n\}$ for which the interarrival times $\{X_n\}$ form an i.i.d. sequence is called a *renewal process*. t_n is then called the n^{th} *renewal epoch* and $F(x) = P(X \le x),\ x \ge 0$, denotes the common interarrival time distribution. The renewal process is simple if $P(X > 0) = 1$, and this ensures that $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ with probability 1. The *rate* of the renewal process is defined as $\lambda \stackrel{\text{def}}{=} 1/E(X)$ which is justified by

Theorem 1.1 (Elementary Renewal Theorem (ERT)) *For a renewal process,*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \text{ w.p.1.}$$

and

$$\lim_{t\to\infty} \frac{E(N(t))}{t} = \lambda.$$

Proof : Observing that $t_{N(t)} \le t < t_{N(t)+1}$ and that $t_{N(t)} = X_1 + \cdots X_{N(t)}$, yields after division by $N(t)$:

$$\frac{1}{N(t)} \sum_{j=1}^{N(t)} X_j \le \frac{t}{N(t)} \le \frac{1}{N(t)} \sum_{j=1}^{N(t)+1} X_j.$$

By the Strong Law of Large Numbers (SLLN), both the left and the right pieces converge to $E(X)$ as $t \longrightarrow \infty$, wp1. Since $t/N(t)$ is sandwiched between the two, it also converges to $E(X)$, yielding the first result after taking reciprocals.

For the second result, we must show that the collection of rvs $\{N(t)/t : t \ge 1\}$ is uniformly integrable (UI)[1], so as to justify the interchange of limit and expected value,

$$\lim_{t\to\infty} \frac{E(N(t))}{t} = E\Big(\lim_{t\to\infty} \frac{N(t)}{t}\Big).$$

We will show that $P(N(t)/t > x) \le c/x^2,\ x > 0$ for some $c > 0$ hence proving UI. To this end, choose $a > 0$ such that $0 < F(a) < 1$ (if no such a exists then the renewal process is deterministic and the result is trival). Define new interarrival times via truncation $\hat{X}_n = aI\{X_n > a\}$. Thus $\hat{X}_n = 0$ with probability $F(a)$ and equals a with probability $1 - F(a)$. Letting $\hat{N}(t)$ denote the counting process obtained by using these new interarrival times, it follows that $N(t) \le \hat{N}(t),\ t \ge 0$. Moreover, arrivals (which now occur in batches) can now only occur at the deterministic lattice of times $\{na : n \ge 0\}$. Letting $p = 1 - F(a)$, and letting K_n denote the number of arrivals that occur at time na, we conclude that

[1] A collection of rvs $\{X_t : t \in T\}$ is said to be uniformly integrable (UI), if $\sup_{t\in T} E(|X_t| I\{|X_t| > x\}) \to 0$, as $x \to \infty$.

$\{K_n\}$ is iid with a geometric distribution with success probability p. Letting $[x]$ denote the smallest integer $\geq x$, we have the inequality

$$N(t) \leq \hat{N}(t) \leq S(t) = \sum_{n=1}^{[t/a]} K_n, \ t \geq 0.$$

Observing that $E(S(t)) = [t/a]E(K)$ and $Var(S(t)) = [t/a]Var(K)$, we obtain $E(S(t)^2) = Var(S(t) + E(S(t))^2 = [t/a]Var(K) + [t/a]^2E^2(K) \leq c_1t + c_2t^2$, for constants $c_1 > 0$, $c_2 > 0$. Finally, when $t \geq 1$, Chebychev's inequality implies that $P(N(t)/t > x) \leq E(N^2(t))/t^2x^2 \leq E(S^2(t))/t^2x^2 \leq c/x^2$ where $c = c_1 + c_2$. ■

Remark 1.1 In the elementary renewal theorem, the case when $\lambda = 0$ (e.g., $E(X) = \infty$) is allowed, in which case the renewal process is said to be *null* recurrent. In the case when $0 < \lambda < \infty$ (e.g., $0 < E(X) < \infty$) the renewal process is said to be *positive* recurrent.

1.3 Poisson point process

There are several equivalent definitions for a Poisson process; we present the simplest one. Although this definition does not indicate why the word "Poisson" is used, that will be made apparent soon. Recall that a renewal process is a point process $\psi = \{t_n : n \geq 0\}$ in which the interarrival times $X_n = t_n - t_{n-1}$ are i.i.d. r.v.s. with common distribution $F(x) = P(X \leq x)$. The arrival rate is given by $\lambda = \{E(X)\}^{-1}$ which is justified by the ERT (Theorem 1.1).

In what follows it helps to imagine that the arrival times t_n correspond to the consecutive times that a subway arrives to your station, and that you are interested in catching the next subway.

Definition 1.2 *A Poisson process at rate* $0 < \lambda < \infty$ *is a renewal point process in which the interarrival time distribution is exponential with rate* λ*: interarrival times* $\{X_n : n \geq 1\}$ *are i.i.d. with common distribution* $F(x) = P(X \leq x) = 1 - e^{-\lambda x}, \ x \geq 0$*;* $E(X) = 1/\lambda$*.*

Since $t_n = X_1 + \cdots + X_n$ (the sum of n i.i.d. exponentially distributed r.v.s.), we conclude that the distribution of t_n is the n^{th}-fold convolution of the exponential distribution and thus is a $gamma(n, \lambda)$ distribution (also called an $Erlang(n, \lambda)$ distribution); its density is given by

$$f_n(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}, \ t \geq 0, \tag{2}$$

where $f_1(t) = f(t) = \lambda e^{-\lambda t}$ is the exponential density, and $E(t_n) = E(X_1 + \cdots + X_n) = nE(X) = n/\lambda$.

For example, f_2 is the convolution $f_1 * f_1$:

$$\begin{aligned} f_2(t) &= \int_0^t f_1(t-s)f_1(s)ds \\ &= \int_0^t \lambda e^{-\lambda(t-s)} ds \lambda e^{-\lambda s} ds \\ &= \lambda e^{-\lambda t} \int_0^t \lambda ds \\ &= \lambda e^{-\lambda t} \lambda t, \end{aligned}$$

and in general $f_{n+1} = f_n * f_1 = f_1 * f_n$.

1.4 The Poisson distribution: A Poisson process has Poisson increments

Later, in Section 1.5 (Proposition 1.1) we will prove the fundamental fact that: For each fixed $t > 0$, the distribution of $N(t)$ is Poisson with mean λt:

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \ k \geq 0.$$

In particular, $E(N(t)) = \lambda t, \ Var(N(t)) = \lambda t, \ t \geq 0$. In fact, the number of arrivals in any arbitrary interval of length t, $N(s+t) - N(s)$ is also Poisson with mean λt:

$$P(N(s+t) - N(s) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \ s > 0, \ k \geq 0,$$

and $E(N(s+t) - N(s)) = \lambda t, \ Var(N(s+t) - N(s)) = \lambda t, \ t \geq 0$.

$N(s+t) - N(s)$ is called a length t *increment* of the counting process $\{N(t) : t \geq 0\}$; the above tells us that the Poisson counting process has increments that have a distribution that is Poisson and only depends on the length of the increment. Any increment of length t is distributed as Poisson with mean λt.

1.5 Stationary and independent increments characterization of the Poisson process

Suppose that subway arrival times to a given station form a Poisson process at rate λ. If you enter the subway station at time $s > 0$ it is natural to consider how long you must wait until the next subway arrives. But $t_{N(s)} \leq s < t_{N(s)+1}$; s lies somewhere within a subway interarrival time. For example if $N(s) = 4$ then $t_4 \leq s < t_5$ and s lies somewhere within the interarrival time $X_5 = t_5 - t_4$. But since the interarrival times have an exponential distribution, they have the memoryless property and thus your waiting time, $A(s) = t_{N(s)+1} - s$, until the next subway, being the remainder of an originally exponential r.v., is itself an exponential r.v. and independent of the past: $P(A(s) > t) = e^{-\lambda t},\ t \geq 0$. Once the next subway arrives (at time $t_{N(s)+1}$), the future interarrival times are i.i.d. exponentials and independent of $A(s)$. But this means that the Poisson process, from time s onword is yet again another Poisson process with the same rate λ; *the Poisson process restarts itself from every time s and is independent of its past.*

In terms of the counting process this means that for fixed $s > 0$, $N(s+t) - N(s)$ (the number of arrivals during the first t time units after time s, the "future") has the same distribution as $N(t)$ (the number of arrivals during the first t time units), and is independent of $\{N(u) : 0 \leq u \leq s\}$ (the counting process up to time s, the "past"). This above discussion illustrates the *stationary* and *independent* increments properties, to be discussed next. It also shows that that $\{N(t) : t \geq 0\}$ is a continuous-time Markov process: The future $\{N(s+t) : t > 0\}$, given the present state $N(s)$, is independent of the past $\{N(u) : 0 \leq u < s\}$. Because the state space is discrete, this is an example of a *Continuous-Time Markov Chain (CTMC)*, a topic that we will cover very soon.

Definition 1.3 *A random point process ψ is said to have stationary increments if for all $t \geq 0$ and all $s \geq 0$ it holds that $N(t+s) - N(s)$ (the number of points in the time interval $(s, s+t]$) has a distribution that only depends on t, the length of the time interval.*

For any interval $I = (a, b]$, let $N(I) = N(b) - N(a)$ denote the number of points that fall in the interval. More generally, for any subset $A \subset \mathbb{R}_+ = [0, \infty)$, let $N(A)$ denote the number of points that fall in the subset A.

Definition 1.4 *ψ is said to have independent increments if for any two non-overlapping intervals of time, I_1 and I_2, the random variables $N(I_1)$ and $N(I_2)$ are independent.*

We conclude from the discussions above that

The Poisson process has both stationary and independent increments.

But what is this distribution of $N(t+s) - N(s)$ that only depends on t, the length of the interval? We now show that it is Poisson for the Poisson process:

Proposition 1.1 *For a Poisson process at rate λ, the distribution of $N(t),\ t > 0$, is Poisson with mean λt:*

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},\ k \geq 0.$$

In particular, $E(N(t)) = \lambda t,\ Var(N(t)) = \lambda t,\ t \geq 0$. Thus by stationary increments, $N(s+t) - N(s)$ is also Poisson with mean λt:

$$P(N(s+t) - N(s) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},\ s > 0,\ k \geq 0,$$

and $E(N(s+t) - N(s)) = \lambda t,\ Var(N(s+t) - N(s)) = \lambda t,\ t \geq 0$.

Proof : Note that $P(N(t) = n) = P(t_n \leq t < t_{n+1}) = P(t_{n+1} > t) - P(t_n > t)$. We will show that

$$P(t_m > t) = e^{-\lambda t}(1 + \lambda t + \cdots + \frac{(\lambda t)^{m-1}}{(m-1)!}),\ m \geq 1, \tag{3}$$

so that substituting in $m = n+1$ and $m = n$ and subtracting yields the result.

To this end, observe that differentiating the tail $Q_n(t) = P(t_n > t)$ (recall that t_n has the $gamma(n, \lambda)$ density in (2)) yields

$$\frac{d}{dt}Q_n(t) = -f_n(t) = -\lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!},\ Q_n(0) = 1.$$

Differentiating the right hand side of (3), we see that (3) is in fact the solution (anti-derivative). ■

Because of the above result, we see that $\lambda = E(N(1))$; the arrival rate is the expected number of arrivals in any length one interval.

Examples

1. Suppose cars arrive to the GW Bridge according to a Poisson process at rate $\lambda = 1000$ per hour. What is the expected value and variance of the number of cars to arrive during the time interval between 2 and 3 o'clock PM?

   $E(N(3) - N(2)) = E(N(1))$ by stationary increments. $E(N(1)) = \lambda 1 = 1000$. Variance is the same, $Var(N(1)) = \lambda 1 = 1000$.

2. (*continuation*)

   What is the expected number of cars to arrive during the time interval between 2 and 3 o'clock PM, given that 700 cars already arrived between 9 and 10 o'clock this morning?

   $E(N(3) - N(2)|N(10) - N(9) = 700) = E(N(3) - N(2)) = E(N(1)) = 1000$ by independent and stationary increments: the r.v.s. $N(I_1) = N(3) - N(2)$ and $N(I_2) = N(10) - N(9)$ are independent.

3. (*continuation*) What is the probability that no cars will arrive during a given 15 minute interval?

   $P(N(0.25) = 0) = e^{-\lambda(0.25)} = e^{-250}$.

Remarkable as it may seem, it turns out that the Poisson process is completely characterized by stationary and independent increments:

Theorem 1.2 *Suppose that ψ is a simple random point process that has both stationary and independent increments. Then in fact, ψ is a Poisson process. Thus the Poisson process is the only simple point process with stationary and independent increments.*

Proof :[Sketch]

Note that

$$\begin{aligned} P(X_1 > s+t) &= P(N(s+t) = 0) \\ &= P(N(s) = 0, N(s+t) - N(s) = 0) \\ &= P(N(s) = 0)P(N(t) = 0) \text{ (via independent and stationary increments)} \\ &= P(X_1 > s)P(X_1 > t), \end{aligned}$$

But this implies that X_1 has the memoryless property, and thus it must be exponentially distributed; $P(X_1 \leq t) = 1 - e^{-\lambda t}$ for some $\lambda > 0$.

But by stationary and independent increments, right after X_1, the counting process $\{N(t)\}$ starts over again and is independent of its past (we are technically using the *strong Markov property* here, $t_1 = X_1$ is a (continuous-time) stopping time for $\{N(t)\}$); thus X_2 is independent of X_1 and has the same distribution. Continuing in this fashion we conclude that the all interarrival times $\{X_n\}$ are iid with an exponential distribution at rate λ.

■

We now have two different ways of determining if a simple point process is a Poisson process: (1) checking if it is a renewal process with an exponential interarrival time distribution, or (2) checking if it has both stationary and independent increments.

1.6 Constructing a Poisson process from independent Bernoulli trials, and the Poisson approximation to the binomial distribution

A Poisson process at rate λ can be viewed as the result of performing an independent Bernoulli trial with success probability $p = \lambda dt$ in each "infinitesimal" time interval of length dt, and placing a point there if the corresponding trial is a success (no point there otherwise). Intuitively, this would yield a point process with both stationary and independent increments; a Poisson process: The number of Bernoulli trials that can be fit in any interval only depends on the length of the interval and thus the distribution for the number of successes in that interval would also only depend on the length; stationary increments follows. For two non-overlapping intervals, the Bernoulli trials in each would be independent of one another since all the trials are i.i.d., thus the number of successes in one interval would be independent of the number of successes in the other interval;

independent increments follows. We proceed next to explain how this Bernoulli trials idea can be made rigorous.

As explained in Lecture Notes on the exponential distribution, the exponential distribution can be obtained as a limit of the geometric distribution: Fix n large, and perform, using success probability $p_n = \lambda/n$, an independent Bernoulli trial at each time point $i/n,\ i \geq 1$. Let Y_n denote the time at which the first success ocurred. Then $Y_n = K_n/n$ where K_n denotes the number of trials until the first success, and has the geometric distribution with success probability p_n; $P(K_n > k) = (1 - \lambda/n)^k,\ k \geq 1$. As $n \to \infty$, Y_n converges to a r.v. Y having the exponential distribution with rate λ $P(Y_n > x) = P(K_n > nx) = (1 - \lambda/n)^{nx} \to e^{-\lambda x}$. This Y thus can serve as the first arrival time t_1 for a Poisson process at rate λ. The idea here is that the tiny intervals of length $1/n$ become (in the limit) the infinitesimal dt intervals. Once we have our first success, at time t_1, we continue onwards in time (in the interval (t_1, ∞)) with new Bernoulli trials until we get the second success at time t_2 and so on until we get all the arrival times $\{t_n : n \geq 1\}$. By construction, each interarrival time, $X_n = t_n - t_{n-1},\ n \geq 1$, is an independent exponentially distributed r.v. with rate λ; hence we constructed a Poisson process at rate λ.

Another key to understanding how the Poisson process can be constructed from Bernoulli trials is the fact that the Poisson distribution can be used to approximate the binomial distribution:

Proposition 1.2 *For $\lambda > 0$ fixed, let $X \sim binomial(n, p)$ with success probability $p_n = \lambda/n$. Then, as $n \to \infty$, X converges in distribution to a Poisson rv with mean λ. Thus, a binomial distribution in which the number of trials n is large and the success probability p is small can be approximated by a Poisson distrbution with mean $\lambda = np$.*

Proof : Since

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k},$$

where $p = \lambda/n$, we must show that for any $k \geq 0$

$$\lim_{n\to\infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}.$$

re-writing and expanding yields

$$\binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k} = \frac{n!}{(n-k)!\ n^k} \times \frac{\lambda^k}{k!} \times \frac{(1 - \lambda/n)^n}{(1 - \lambda/n)^k},$$

the product of three pieces.

But $\lim_{n\to\infty}(1-\lambda/n)^k = 1$ since k is fixed, and from calculus, $\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}$. (Recall a proof via taking natural logarithms and then using L'Hosptal's Rule.) Moreover,

$$\frac{n!}{(n-k)!\ n^k} = \frac{n}{n} \times \frac{(n-1)}{n} \times \cdots \times \frac{(n-k+1)}{n},$$

(the product of k (fixed) pieces) and hence converges to 1 as $n \to \infty$ since each of the k ratios does so. Combining these facts yields the result.

$\blacksquare$

We can use the above result to construct the counting process at time t, $N(t)$, for a Poisson process as follows: Fix $t > 0$. Divide the interval $(0, t]$ into n subintervals, $((i-1)t/n, it/n]$, $1 \le i \le n$, of the equal length t/n. At the right endpoint it/n of each such subinterval, perform a Bernoulli trial with success probability $p_n = \lambda t/n$, and place a point there if successful (no point otherwise). Let $N_n(t)$ denote the number of such points placed (successes). Then $N_n(t) \sim binomial(n, p_n)$ and converges in distribution to $N(t) \sim Poisson(\lambda t)$, as $n \to \infty$. Moreover, the points placed in $(0, t]$ from the Bernoulli trials converge (as $n \to \infty$) to the points $t_1, \ldots, t_{N(t)}$ of the Poisson process during $(0, t]$. So we have actually obtained the Poisson process up to time t.

1.7 Little $o(t)$ results for stationary point processes

Let $o(t)$ denote any function of t that satisfies $o(t)/t \to 0$ as $t \to 0$. Examples include $o(t) = t^n$, $n > 1$, but there are many others.

If ψ is any point process with stationary increments and $\lambda = E(N(1)) < \infty$, then (see below for a discussion of proofs)

$$P(N(t) > 0) = \lambda t + o(t), \tag{4}$$
$$P(N(t) > 1) = o(t). \tag{5}$$

Because of stationary increments we get the same results for any increment of length t, $N(s+t) - N(s)$, and in words (4) can be expressed as

$$P(\text{at least 1 arrival in any interval of length } t) = \lambda t + o(t),$$

whereas (5) can be expressed as

$$P(\text{more than 1 arrival in any interval of length } t) = o(t).$$

Since $P(N(t) = 1) = P(N(t) > 0) - P(N(t) > 1)$, (4) and (5) together yield

$$P(N(t) = 1) = \lambda t + o(t), \tag{6}$$

or in words

$$P(\text{ An arrival in any interval of length } t) = \lambda t + o(t).$$

We thus get for any $s \ge 0$:

$$P(N(s+t) - N(s) = 1) \approx \lambda t, \text{ for } t \text{ small},$$

which using infinitesimals can be written as

$$P(N(s+dt) - N(s) = 1) = \lambda dt. \tag{7}$$

The above $o(t)$ results hold for any (simple) point process with stationary increments, not just a Poisson process. But note how (7) agrees with our Bernoulli trials interpretation of the Poisson process, e.g., performing in each interval of length dt an independent Bernoulli trial with success probability $p = \lambda dt$. But the crucial difference is that our Bernoulli trials construction also yields the independent increments property which is not expressed in (7). This difference helps explain why the Poisson process is the unique simple point process with both stationay *and* independent increments: There are numerous examples of point processes with stationary increments (we shall offer some examples later), but only one with both stationary and independent increments; the Poisson process.

Although a general proof of (4) and (5) is beyond the scope of this course, we will be satisfied with proving it for the Poisson process at rate λ for which it follows directly from the Taylor's expansion for e^x:

$$\begin{aligned}
P(N(t) > 0) &= 1 - e^{-\lambda t} \\
&= 1 - (1 - \lambda t + \frac{(\lambda t)^2}{2} + \cdots) \\
&= \lambda t + \frac{(\lambda t)^2}{2} + \cdots) \\
&= \lambda t + o(t).
\end{aligned}$$

$$\begin{aligned}
P(N(t) > 1) &= P(N(t) = 2) + P(N(t) = 3) + \cdots \\
&= e^{-\lambda t}(\frac{(\lambda t)^2}{2} + \cdots) \\
&\leq (\frac{(\lambda t)^2}{2} + \cdots) \\
&= o(t)
\end{aligned}$$

1.8 Partitioning Theorems for Poisson processes and random variables

Given $X \sim Poiss(\alpha)$ (a Poisson rv with mean α) suppose that we imagine that X denotes some number of objects (arrivals during some fixed time interval for example), and that independent of one another, each such object is of type 1 or type 2 with probability p and $q = 1 - p$ respectively. This means that if $X = n$ then the number of those n that are of type 1 has a $binomial(n, p)$ distribution and the number of those n that are of type 2 has a $binomial(n, q)$ distribution. Let X_1 and X_2 denote the number of type 1 and type 2 objects respectively ; $X_1 + X_2 = X$. The following shows that in fact if we do this, then X_1 and X_2 are independent Poisson random variables with means $p\alpha$ and $q\alpha$ respectively.

Theorem 1.3 (Partitioning a Poisson r.v.) *If $X \sim Poiss(\alpha)$ and if each object of X is, independently, type 1 or type 2 with probability p and $q = 1 - p$, then in fact $X_1 \sim Poiss(p\alpha)$, $X_2 \sim Poiss(q\alpha)$ and they are independent.*

Proof : We must show that

$$P(X_1 = k, X_2 = m) = e^{-p\alpha}\frac{(p\alpha)^k}{k!}\; e^{-q\alpha}\frac{(q\alpha)^m}{m!}. \tag{8}$$

$$P(X_1 = k, X_2 = m) = P(X_1 = k, X = k + m) = P(X_1 = k|X = k + m)P(X = k + m).$$

But given $X = k + m$, $X_1 \sim Bin(k + m, p)$ yielding

$$P(X_1 = k|X = k + m)P(X = k + m) = \frac{(k+m)!}{k!m!}p^k q^m e^{\alpha}\frac{\alpha^{k+m}}{(k+m)!}.$$

Using the fact that $e^{\alpha} = e^{p\alpha}e^{q\alpha}$ and other similar algabraic identites, the above reduces to (8) as was to be shown. ∎

The above theorem generalizes to Poisson processes:

Theorem 1.4 (Partitioning a Poisson process) *If $\psi \sim PP(\lambda)$ and if each arrival of ψ is, independently, type 1 or type 2 with probability p and $q = 1 - p$ then in fact, letting ψ_i denote the point process of type i arrivals, $i = 1, 2$, $\psi_1 \sim PP(p\lambda)$, $\psi_2 \sim PP(q\lambda)$ and they are independent.*

Proof : It is immediate that each ψ_i is a Poisson process since each remains having stationary and independent increments. Let $N(t)$ and $N_i(t)$, $i = 1, 2$ denote the corresponding counting processes, $N(t) = N_1(t) + N_2(t)$, $t \geq 0$. From Theorem 1.3, $N_1(1)$ and $N_2(1)$ are independent Poisson rvs with means $E(N_1(1)) = \lambda_1 = p\lambda$ and $E(N_1(1)) = \lambda_2 = q\lambda$ since they are a partitioning of $N(1)$; thus ψ_i indeed has rate λ_i, $i = 1, 2$. What remains to show is that the two counting processes (as processes) are independent. But this is immediate from Theorem 1.3 and independent increments of ψ: If we take any collection of non-overlapping intervals (sets more generally) $A_1, \ldots A_k$ and non-overlapping intervals $B_1, \ldots B_l$ then we must show that $(N_1(A_1), \ldots, N_1(A_k))$ is independent of $(N_2(B_1), \ldots, N_2(B_l))$ argued as follows: Any part (say subset C) of the A_i which intersect with the B_i will yield independence due to partitioning of the rv $N(C)$, and any parts of the A_i that are disjoint from the B_i will yield independence due to the independent increments of ψ; thus independence follows. ∎

The above is quite interesting for it means that if Poisson arrivals at rate λ come to our lecture room, and upon each arrival we flip a coin (having probability p of landing heads), and route all those for which the coin lands tails (type 2) into a different room, only allowing those for which the coin lands heads (type 1) enter our room, then the arrival processes to the two rooms are independent and Poisson.

For example, suppose that $\lambda = 30$ per hour, and $p = 0.6$. Letting $N_1(t)$ and $N_2(t)$ denote the counting processes for type 1 and type 2 respectively, this means that $N_1(t) \sim Poiss(\alpha)$ where $\alpha = (0.6)(30)(t) = 18t$. Now consider the two events

$$A = \{5 \text{ arrivals into room 1 during the hours 1 to 3}\}$$

and

$$B = \{1000 \text{ arrivals into room 2 during the hours 1 to 3}\}.$$

We thus conclude that the two events A and B are independent yielding

$$\begin{aligned} P(A|B) &= P(A) \\ &= P(N_1(3) - N_1(1) = 5) \\ &= P(N_1(2) = 5) \\ &= e^{-36}\frac{36^5}{5!}. \end{aligned}$$

In the above computation, the third equality follows from stationary increments (of type 1 arrivals since they are Poisson at rate 18).

1.8.1 Superposition of independent Poisson processes

In the previous section we saw that a Poisson process ψ can be partitioned into two independent ones ψ_1 and ψ_2 (type 1 and type 2 arrivals). But this means that they can be put back together again to obtain ψ. Putting together means taking the *superposition* of the two point processes, that is, combining all their points together, then placing them in acsending order, to form one point process ψ (regardless of type). We write this as $\psi = \psi_1 + \psi_2$, and of course we in particular have $N(t) = N_1(t) + N_2(t),\ t \geq 0$.

A little thought reveals that therefore we can in fact start with any two independent Poisson processes, $\psi_1 \sim PP(\lambda_1)$, $\psi_2 \sim PP(\lambda_2)$ (call them type 1 and type 2) and superpose them to obtain a Poisson process $\psi = \psi_1 + \psi_2$ at rate $\lambda = \lambda_1 + \lambda_2$. The partitioning probability p is given by

$$p = \frac{\lambda_1}{\lambda_1 + \lambda_2},$$

because that is the required p which would allow us to partition a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$ into two independent Poisson processes at rate λ_1 and λ_2; $\lambda p = \lambda_1$ and $\lambda(1-p) = \lambda_2$ as is required. p is simply the probability that (starting from any time t) the next arrival time of type 1 (call this Y_1) ocurrs before the next arrival time of type 2 (call this Y_2), which by the memoryless proberty is given by $P(Y_1 < Y_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$ because $Y_1 \sim exp(\lambda_1),\ Y_2 \sim exp(\lambda_2)$ and they are independent by assumption. Once an arrival ocurrs, the memoryless property allows us to conclude that the next arrival will yet again be of type 1 or 2 (independent of the past) depending only on which of two independent exponentialy distributed r.v.s. is smaller; $P(Y_1 < Y_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$. Continuing in this fashion we conclude that indeed each arrival from the superposition ψ is, independent of all others, of type 1 or type 2 with probability $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

Arguing directly that the superposition of independent Poisson processes yields a Poisson process is easy: The superposition has both stationary and independent increments, and thus must be a Poisson process. Moreover $E(N(1)) = E(N_1(1))+E(N_2(1)) = \lambda_1+\lambda_2$, so the rate indeed is given by $\lambda = \lambda_1 + \lambda_2$.

Examples

Foreign phone calls are made to your home phone according to a Poisson process at rate $\lambda_1 = 2$ (per hour). Independently, domestic phone calls are made to your home phone according to a Poisson process at rate $\lambda_2 = 5$ (per hour).

1. You arrive home. What is the probability that the next call will be foreign? That the next three calls will be domestic?

   Answers: $\frac{\lambda_1}{\lambda_1+\lambda_2} = 2/7$, $(\frac{\lambda_2}{\lambda_1+\lambda_2})^3 = (5/7)^3$. Once a domestic call comes in, the future is independent of the past and has the same distribution as when we started by the memoryless property, so the next call will, once again be domestic with the same probability 5/7 and so on.

2. You leave home for 2 hours. What is the mean and variance of the number of calls you recieved during your absence?

   Answer: The superposition of the two types is a Poisson process at rate $\lambda = \lambda_1 + \lambda_2 = 7$. Letting $N(t)$ denote the number of calls by time t, it follows that $N(t)$ has a Poisson distribution with parameter λt; $E(N(2)) = 2\lambda = 14 = Var(N(2))$.

3. Given that there were exactly 5 calls in a given 4 hour period, what is the probability that exactly 2 of them were foreign?

   Answer: The superposition of the two types is a Poisson process at rate $\lambda = \lambda_1 + \lambda_2 = 7$. The individual foriegn and domestic arrival processes can be viewed as type 1 and 2 of a partitioning with $p = \frac{\lambda_1}{\lambda_1+\lambda_2} = 2/7$. Thus given $N(4) = 5$, the number of those 5 that are foriegn (type 1) has a $Bin(5, p)$ distribution. with $p = 2/7$. Thus we want

   $$\binom{5}{2} p^2(1-p)^3.$$

1.9 Constructing a Poisson process up to time t by using the order statistics of iid uniform rvs

Suppose that for a Poisson process at rate λ, we condition on the event $\{N(t) = 1\}$, the event that exactly one arrival ocurred during $(0, t]$. We might conjecture that under such conditioning, t_1 should be uniformly distributed over $(0, t)$. To see that this is in fact so, choose $s \in (0, t)$. Then

$$\begin{aligned} P(t_1 \le s | N(t) = 1) &= \frac{P(t_1 \le s, N(t) = 1)}{P(N(t) = 1)} \\ &= \frac{P(N(s) = 1, N(t) - N(s) = 0)}{P(N(t) = 1)} \\ &= \frac{e^{-\lambda s} \lambda s e^{-\lambda(t-s)}}{e^{-\lambda t} \lambda t} \\ &= \frac{s}{t}. \end{aligned}$$

It turns out that this result generalizes nicely to any number of arrivals, and we present this next.

Let $V_1, V_2, \ldots, V_n$ be n i.i.d uniformly distributed r.v.s. on the interval $(0, t)$. Let $V_{(1)} < V_{(2)} < \cdots < V_{(n)}$ denote them placed in ascending order. Thus $V_{(1)}$ is the smallest of them , $V_{(2)}$ the second smallest and finally $V_{(n)}$ is the largest one. $V_{(i)}$ is called the i^{th} *order statistic* of $V_1, \ldots V_n$.

Note that the joint density function of $(V_1, V_2, \ldots, V_n)$ is given by

$$g(s_1, s_2, \ldots, s_n) = \frac{1}{t^n}, \; s_i \in (0, t),$$

because each V_i has density function $1/t$ and they are independent. Now given any ascending sequence $0 < s_1 < s_2 < \cdots < s_n < t$ it follows that the joint distribution $f(s_1, s_2, \ldots, s_n)$ of the order statistics $(V_{(1)}, V_{(2)}, \ldots, V_{(n)})$ is given by

$$f(s_1, s_2, \ldots, s_n) = \frac{n!}{t^n},$$

because there are $n!$ permutations of the sequence $(s_1, s_2, \ldots, s_n)$ each of which leads to the same order statistics. For example if $(s_1, s_2, s_3) = (1, 2, 3)$ then there are $3! = 6$ permutations all yielding the same order statistics $(1, 2, 3)$: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$.

Theorem 1.5 *For any Poisson process, conditional on the event $\{N(t) = n\}$, the joint distribution of the n arrival times $t_1, \ldots, t_n$ is the same as the joint distribution of $V_{(1)}, \ldots, V_{(n)}$, the order statistics of n i.i.d. unif$(0, t)$ r.v.s., that is, it is given by*

$$f(s_1, s_2, \ldots, s_n) = \frac{n!}{t^n}, \; 0 < s_1 < s_2 < \cdots < s_n < t$$

Proof : We will compute the density for

$$P(t_1 = s_1, \ldots, t_n = s_n | N(t) = n) = \frac{P(t_1 = s_1, \ldots, t_n = s_n, N(t) = n)}{P(N(t) = n)}$$

and see that it is precisely $\frac{n!}{t^n}$. To this end, we re-write the event $\{t_1 = s_1, \ldots, t_n = s_n, N(t) = n\}$ in terms of the i.i.d. interarrival times as $\{X_1 = s_1, \ldots, X_n = s_n - s_{n-1}, X_{n+1} > t - s_n\}$. For example if $N(t) = 2$, then $\{t_1 = s_1, t_2 = s_2, N(t) = 2\} = \{X_1 = s_1, X_2 = s_2 - s_1, X_3 > t - s_2\}$ and thus has density $\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} e^{-\lambda(t-s_2)}) = \lambda^2 e^{-\lambda t}$ due to the independence of the r.v.s. X_1, X_2, X_3, and the algebraic cancellations in the exponents.

We conclude that

$$\begin{aligned} P(t_1 = s_1, \ldots, t_n = s_n | N(t) = n) &= \frac{P(t_1 = s_1, \ldots, t_n = s_n, N(t) = n)}{P(N(t) = n)} \\ &= \frac{P(X_1 = s_1, \ldots, X_n = s_n - s_{n-1}, X_{n+1} > t - s_n)}{P(N(t) = n)} \\ &= \frac{\lambda^n e^{-\lambda t}}{P(N(t) = n)} \\ &= \frac{n!}{t^n}, \end{aligned}$$

where the last equality follows since $P(N(t) = n) = e^{-\lambda t}(\lambda t)^n / n!$.

■

One nice consequence of the above Theorem: If you want to simulate a Poisson process up to time t, you need only first simulate the value of $N(t)$ (a Poisson rv with mean λt, use, for example the discrete inverse transform method), then if $N(t) = n$ generate n i.i.d. $Unif(0, t)$ numbers $(V_1, V_2, \ldots, V_n)$, place them in ascending order $(V_{(1)}, V_{(2)}, \ldots, V_{(n)})$ and finally define $t_i = V_{(i)}$, $1 \le i \le n$.

Uniform numbers are very easy to generate on a computer and so this method can have computational advantages over simply generating exponential r.v.s. for interarrival times X_n, and defining $t_n = X_1 + \cdots + X_n$. Exponential r.v.s. require taking logarithms to generate:

$$X_i = -\frac{1}{\lambda} log(U_i),$$

where $U_i \sim Unif(0, 1)$ and this can be computationally time consuming.

1.10 Applications

1. A bus platform is now empty of passengers, and the next bus will depart in t minutes. Passengers arrive to the platform according to a Poisson process at rate λ. What is the average waiting time of an arriving passenger?

Answer: Let $\{N(t)\}$ denote the counting process for passenger arrivals. Given $N(t) = n \geq 1$, we can treat the n passenger arrival times $t_1, \ldots, t_n$ as the order statistics $V_{(1)} < V_{(2)} < \cdots < V_{(n)}$ of n independent $unif(0,t)$ r.v.s., $V_1, V_2, \ldots, V_n$.

We thus expect that on average a customer waits $E(V) = t/2$ units of time. This indeed is so, proven as follows: The i^{th} arrival has waiting time $W_i = t - t_i$, and there will be $N(t)$ such arrivals. Thus we need to compute $E(T)$ where

$$T = \frac{1}{N(t)} \sum_{i=1}^{N(t)} (t - t_i).$$

(We only consider the case when $N(t) \geq 1$.)

But given $N(t) = n$, we conclude that

$$\begin{aligned} T &= \frac{1}{n} \sum_{i=1}^{n} (t - V_{(i)}) \\ &= \frac{1}{n} \sum_{i=1}^{n} (t - V_i), \end{aligned}$$

because the sum of all n of the $V_{(i)}$ is the same as the sum of all n of the V_i. Thus

$$E(T|N(t) = n)) = \frac{1}{n} nE(t - V) = E(t - V) = \frac{t}{2}.$$

This being true for all $n \geq 1$, we conclude that $E(T) = \frac{t}{2}$.

2. *$M/GI/\infty$ queue:* Arrival times t_n form a Poisson process at rate λ with counting process $\{N(t)\}$, service times S_n are i.i.d. with general distribution $G(x) = P(S \leq x)$ and mean $1/\mu$. There are an infinite number of servers and so there is no delay: the n^{th} customer arrives at time t_n, enters service immediately at any free server and then departs at time $t_n + S_n$; S_n is the length of time the customer spends in service. Let $X(t)$ denote the number of customers in service at time t. We assume that $X(0) = 0$.

   We will now show that

   Proposition 1.3 *For every fixed $t > 0$, the distribution of $X(t)$ is Poisson with parameter $\alpha(t)$ where*

   $$\alpha(t) = \lambda \int_0^t P(S > x)dx,$$

   $$P(X(t) = n) = e^{-\alpha(t)} \frac{(\alpha(t))^n}{n!}, \; n \geq 0.$$

Thus $E(X(t) = \alpha(t) = Var(X(t))$. *Moreover since* $\alpha(t)$ *converges (as* $t \to \infty$*) to*

$$\lambda \int_0^\infty P(S > x)dx = \lambda E(S) = \frac{\lambda}{\mu},$$

we conclude that

$$\lim_{t\to\infty} P(X(t) = n) = e^{-\rho}\frac{\rho^n}{n!},\ n \geq 0,$$

where $\rho = \lambda/\mu$. *So the limiting (or steady-state, or stationary) distribution of* $X(t)$ *exists and is Poisson with parameter* ρ.

Proof : The method of proof is actually based on partitioning the Poisson random variable $N(t)$ into two types: those that are in service at time t, $X(t)$, and those that have departed by time t (denoted by $D(t)$). Thus we need only figure out what is the probability $p(t)$ that a customer who arrived during $(0, t]$ (that is, one of the $N(t)$ arrivals) is still in service at time t.

We first recall that conditional on $N(t) = n$ the n (unordered) arrival times are i.i.d. with a $Unif(0, t)$ distribution. Letting V denote a typical such arrival time, and S their service time, we conclude that this customer will still be in service at time t if and only if $V + S > t$ (arival time + service time $> t$); $S > t - V$. Thus $p(t) = P(S > t - V)$, where S and V are assumed independent. But (as is easily shown)) $t-V \sim Unif(0, t)$ if $V \sim Unif(0, t)$. Thus $p = P(S > t-V) = P(S > V)$. Noting that $P(S > V|V = x) = P(S > x)$ we conclude that

$$p(t) = P(S > V) = \frac{1}{t}\int_0^t P(S > x)dx,$$

where we have conditioned on the value of $V = x \in (0, t)$ (with density $1/t$) and integrated over all such values. This did not depend upon the value n and so we are done.

Thus for fixed t, we can partition $N(t)$ into two independent Poisson r.v.s., $X(t)$ and $D(t)$ (the number of departures by t), to conclude that $X(t) \sim Poiss(\alpha(t))$ where

$$\alpha(t) = \lambda t p(t) = \lambda \int_0^t P(S > x)dx$$

as was to be shown. Similarly $D(t) \sim Poisson(\beta(t))$ where $\beta(t) = \lambda t(1-p(t))$. ■

Recall that

$$p_j \stackrel{\text{def}}{=} \lim_{t\to\infty} P(X(t) = j) = e^{-\rho}\frac{\rho^j}{j!},$$

that is, that the *limiting* (or steady-state or stationary) distribution of $X(t)$ is Poisson with mean ρ. Keep in mind that this implies convergence in a time average sense also:

$$p_j = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(X(s) = j)ds = e^{-\rho}\frac{\rho^j}{j!},$$

which is exactly the continuous time analog of the stationary distribution π for Markov chains:

$$\pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P(X_k = j).$$

Thus we interpret p_j as the long run proportion of time that there are j busy servers. The average number of busy severs is given by the mean of the limiting distribution:

$$L = \sum_{j=0}^{\infty} jp_j = \rho.$$

Finally note that the mean ρ agrees with our "Little's Law" ($L = \lambda w$) derivation of the time average number in system:

$$L = \lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds = \rho.$$

Customers arrival at rate λ and have an average sojourn time of $1/\mu$ yielding $L = \rho$. In short, the time average number in system is equal to the mean of the limiting distribution $\{p_j\}$ for number in system.

Final comment: In proving Proposition 1.3, as with the Example 1, we must use the fact that summing up over all the order statistics is the same as summing up over the non ordered uniforms. When $N(t) = n$, we have

$$X(t) = \sum_{j=1}^{n} I\{S_j > t - V_{(j)}\}.$$

But since service times are i.i.d. and independent of the uniforms, we see that this is the same in distribution as the sum

$$\sum_{j=1}^{n} I\{S_j > t - V_j\}.$$

And since since the $t-V_j$ are are also iid uniform over $(0,t)$, this simplifies in distribution to

$$\sum_{j=1}^{n} I\{S_j > V_j\}.$$

Since the $I\{S_j > V_j\}$ are i.i.d. *Bernoulli*$(p(t))$ r.v.s. with $p(t) = P(S > V) = \frac{1}{t}\int_0^t P(S > u)du$, we conclude that the conditional distribution of $X(t)$ given that $N(t) = n$, is binomial with success probability $p(t)$. Thus $X(t)$ indeed can be viewed as a partitioned $N(t)$ with partition probability $p(t) = P(S > V)$.

Example

Suppose people in NYC buy advance tickets to a movie according to a Poisson process at rate $\lambda = 500$ (per day), and that each buyer independent of all others keeps the ticket (before using) for an amount of time that is distributed as $G(x) = P(S \le x) = x/4,\ x \in (0,4)$ (days), the uniform distribution over $(0,4)$. Assuming that no one owns a ticket at time $t = 0$, what is the expected number of ticket holders at time $t = 2$ days? 5 days?, 5 years?

Answer: we want $E(X(2)) = \alpha(2)$ and $E(X(5)) = \alpha(5)$ and $E(X(5 \times 360)) = \alpha(1800)$ for the $M/G/\infty$ queue in which

$$\alpha(t) = 500 \int_0^t P(S > x)dx.$$

Here $P(S > x) = 1 - x/4,\ x \in (0,4)$ but $P(S > x) = 0,\ x \ge 4$. Thus

$$\alpha(2) = 500 \int_0^2 (1 - x/4)dx = 500(3/2) = 750,$$

and

$$\alpha(5) = \alpha(1800)) = 500 \int_0^4 (1 - x/4)dx = 500E(S) = 500(2) = 1000.$$

The point here is that $\alpha(t) = \lambda E(S) = \rho = 1000,\ t \ge 4$: From time $t = 4$ (days) onwards, the limiting distribution is already reached (no need to take the limit $t \to \infty$). It is Poisson with mean $\rho = 1000$; the distribution of $X(t)$ at time $t = 5$ days is the same as at time $t = 5$ years.

If S has an exponential distribution with mean 2 (days), $P(S > x) = e^{-.5x},\ x \ge 0$, then the answers to the above questions would change. In this case

$$\alpha(t) = 500 \int_0^t e^{-.5x}dx = 1000(1 - e^{-.5t}),\ t \ge 0.$$

The limiting distribution is the same, (Poisson with mean $\rho = 1000$) but we need to take the limit $t \to \infty$ to reach it.

Finally note that if (say) S has a uniform distribution over an interval (a, b) with $a > 0$, then $P(S > x) = 1,\ x \in [0, a)$ and we must use that fact when computing $\alpha(t)$. For example, suppose S is uniform over $(1, 3)$, and we want to compute $\alpha(2)$.

Then $\alpha(2) = \lambda \int_0^1 1dx + \lambda \int_1^2 (3 - x)/2dx =$
$\lambda + \lambda \int_1^2 (3 - x)/2dx =$
$\lambda(1 + 3/4)$.