1 Stopping Times

1.1 Stopping Times: Definition

Given a stochastic process $\mathbf{X} = \{X_n : n \geq 0\}$, a *random time* τ is a discrete random variable on the same probability space as $\mathbf{X}$, taking values in the time set $\mathbb{N} = \{0, 1, 2, \ldots\}$. X_τ denotes the state at the random time τ; if $\tau = n$, then $X_\tau = X_n$. If we were to observe the values $X_0, X_1, \ldots$, sequentially in time and then "stop" doing so right after some time n, basing our decision to stop on (at most) only what we have seen thus far, then we have the essence of a *stopping time*. The basic feature is that we do not know the future hence can't base our decision to stop now on knowing the future. But we also are not allowed to consider other information that is correlated with $\mathbf{X}$. To make this precise, let the *total information known up to time n*, for any given $n \geq 0$, be defined as all the information (events) contained in $\{X_0, \ldots, X_n\}$. For example, events of the form $\{X_0 \in A_0, X_1 \in A_1, \ldots, X_n \in A_n\}$, where the $A_i \subset \mathcal{S}$ are subsets of the state space.

Definition 1.1 *Let $\mathbf{X} = \{X_n : n \geq 0\}$ be a stochastic process. A stopping time with respect to $\mathbf{X}$ is a random time such that for each $n \geq 0$, the event $\{\tau = n\}$ is completely determined by (at most) the total information known up to time n, $\{X_0, \ldots, X_n\}$. (It can, however, depend on other information that is independent of $\mathbf{X}$, such as the outcome of an independent coin flip.)*

In the context of gambling, in which X_n denotes our total earnings after the n^{th} gamble, a stopping time τ is thus a rule that tells us at what time to stop gambling. Our decision to stop after a given gamble can only depend (at most) on the "information" known at that time.

If X_n denotes the price of a stock at time n and τ denotes the time at which we will sell the stock, then our decision to sell the stock at a given stopping time can only depend on the "information" known at that time. The optimal time at which one might exercise an American style option is yet again another example.

Remark 1 All of this can be defined analogously for a sequence $\{X_1, X_2, \ldots\}$ in which time is strictly positive; $n = 1, 2, \ldots$.: τ is a stopping time with respect to this sequence if $\{\tau = n\}$ is completely determined by (at most) the total information known up to time n, $\{X_1, \ldots, X_n\}$.

1.2 Examples

1. *(First passage/hitting times/Gambler's ruin problem:)* Suppose that $\mathbf{X}$ has a discrete state space and let i be a fixed state. Let

   $$\tau = \min\{n \geq 0 : X_n = j\},$$

   the first time that the process visits state j. This is called the first passage time of the process into state j, also called the *hitting time* of the process to state j. More generally we can let A be a collection of states such as $A = \{2, 3, 9\}$ or A=$\{2, 4, 6, 8, \ldots\}$, and then τ is the first passage time (hitting time) into the set A:

   $$\tau = \min\{n \geq 0 : X_n \in A\}.$$

   As a special case we can consider the time at which the gambler stops in the gambler's ruin problem; the gambling stops when either $X_n = N$ or $X_n = 0$ whichever happens first; the first passage time to the set $A = \{0, N\}$. This represents the number of gambles until the game ends.

Proving that hitting times are stopping times is simple:

$\{\tau = 0\} = \{X_0 \in A\}$, hence only depends on X_0, and for $n \geq 1$,

$$\{\tau = n\} = \{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A\},$$

and thus only depends on $\{X_0, \ldots, X_n\}$ as is required. When $A = \{j\}$ this reduces to the hitting time to state j and

$$\{\tau = n\} = \{X_0 \neq j, \ldots, X_{n-1} \neq j, X_n = j\}.$$

In the gambler's ruin problem with $X_0 = i \in \{1, \ldots, N-1\}$, $X_n = i + \Delta_1 + \cdots + \Delta_n$ (simple random walk) until the set $A = \{0, N\}$ is hit. Thus $\tau = \min\{n \geq 0 : X_n \in A\}$ is a stopping time with respect to both $\{X_n\}$ and $\{\Delta_n\}$. For example, for $n \geq 2$,

$$\begin{aligned}\{\tau = n\} &= \{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A\}, \\ &= \{i + \Delta_1 \notin A, \ldots, i + \Delta_1 + \cdots + \Delta_{n-1} \notin A, i + \Delta_1 + \cdots + \Delta_n \in A\},\end{aligned}$$

and thus is completely determined by both $\{X_0, \ldots, X_n\}$ and $\{\Delta_1, \ldots, \Delta_n\}$. The point here is that if we know the intial condition $X_0 = i$, then $\{X_n\}$ and $\{\Delta_n\}$ contain the same information.

2. *(Independent case)* Let $\mathbf{X} = \{X_n : n \geq 0\}$ be any stochastic process and suppose that τ is any random time that is independent of $\mathbf{X}$. Then τ is a stopping time. In this case, $\{\tau = n\}$ doesn't depend *at all* on $\mathbf{X}$ (past or future); it is independent of it. An example might be: Before you begin gambling you decide that you will stop gambling after the 10th gamble (regardless of all else). In this case $P(\tau = 10) = 1$. Another example: Every day after looking at the stock price, you flip a coin. You decide to sell the stock the first time that the coin lands heads. (I do not recommend doing this!) In this case τ is independent of the stock pricing and has a geometric distribution.

3. *(Example of a non-stopping time: Last exit time)* Consider the rat in the open maze problem in which the rat eventually reaches freedom (state 0) and never returns into the maze. Assume the rat starts off in cell 1; $X_0 = 1$. Let τ denote the *last time* that the rat visits cell 1 before leaving the maze:

$$\tau = \max\{n \geq 0 : X_n = 1\}.$$

Clearly we need to know the future to determine such a time. For example the event $\{\tau = 0\}$ tells us that in fact the rat never returned to state 1: $\{\tau = 0\} = \{X_0 = 1, X_1 \neq 1, X_2 \neq 1, X_3 \neq 1, \ldots\}$. Clearly this depends on all of the future, not just X_0. Thus this is not a stopping time.

In general a last exit time (the last time that a process hits a given state or set of states) is not a stopping time; in order to know that the last visit has just occurred, one must know the future.

4. *(Another example of a non-stopping time: correlated information:)* Consider the sequential price/share of a stock I at the end of the n^{th} day, X_n. Now let Y_n denote the price/share of yet another stock II that is correlated with I but is different. Let

$$\tau = \min\{n \geq 0 : X_n > 25, \ Y_n > 35\}.$$

Then the event $\{\tau = n\}$ depends on the joint information $\{X_0, Y_0, \ldots, X_n, Y_n\}$ not just the allowable $\{X_1, \ldots, X_n\}$. We could say that τ is a stopping time with respect to the joint process $(\mathbf{X}, \mathbf{Y})$, but it is not a stopping time with respect to $\mathbf{X}$ alone.

1.3 Other formulations for stopping time

If τ is a stopping time with respect to $\{X_n\}$, then we conclude that the event $\{\tau \le n\}$ can only depend at most on $\{X_0, \ldots, X_n\}$ as well: stopping by time n can only depend on the information up to time n. Formally we can prove this as follows: $\{\tau \le n\}$ is the union of events

$$\{\tau \le n\} = \cup_{j=0}^{n} \{\tau = j\}.$$

By the definition of stopping time, each $\{\tau = j\}$, $j \le n$, depends (at most) on $\{X_0, \ldots, X_j\}$ which is contained in $\{X_0, \ldots, X_n\}$. Thus the union is also contained in $\{X_0, \ldots, X_n\}$.

Similarly we can handle a set like $\{\tau < n\}$ since we can re-write it as $\{\tau \le n-1\}$; thus it is determined by $\{X_0, \ldots, X_{n-1}\}$. Also, we can handle a set like $\{\tau > n\}$, since it is equivalent to $\overline{\{\tau \le n\}}$, denoting the complement of the event $\{\tau \le n\}$: since $\{\tau \le n\}$ is determined by $\{X_0, \ldots, X_n\}$, so is its complement. For example, if $\tau = \min\{n \ge 0 : X_n \in A\}$, a hitting time, then $\{\tau > n\} = \{X_0 \notin A, X_1 \notin A, \ldots, X_n \notin A\}$, and hence only depends on $\{X_0, \ldots, X_n\}$.

1.4 Wald's Equation

We now consider the very special case of stopping times when $\{X_n : n \ge 1\}$ is an independent and identically distributed (i.i.d.) sequence with common mean $E(X)$. We are interested in the sum of the r.v.s. up to time τ,

$$\sum_{n=1}^{\tau} X_n = X_1 + \cdots + X_\tau.$$

If $\tau = k$, then $\sum_{n=1}^{\tau} X_n = \sum_{n=1}^{k} X_n$.

Theorem 1.1 (Wald's Equation) *If τ is a stopping time with respect to an i.i.d. sequence $\{X_n : n \ge 1\}$, and if $E(\tau) < \infty$ and $E(|X|) < \infty$, then*

$$E\{\sum_{n=1}^{\tau} X_n\} = E(\tau)E(X).$$

Before we prove this, note that this is a generalization of the fact that for any fixed integer $k \ge 1$,

$$E(X_1 + \cdots + X_k) = kE(X).$$

Wald's equation allows us to replace deterministic time k by the expected value of a random time τ when τ is a stopping time.

Proof :

$$\sum_{n=1}^{\tau} X_n = \sum_{n=1}^{\infty} X_n I\{\tau \ge n\} = \sum_{n=1}^{\infty} X_n I\{\tau > n-1\}, \tag{1}$$

where $I\{\tau > n-1\}$ denotes the indicator r.v. for the event $\{\tau > n-1\}$. By the definition of stopping time, $\{\tau > n-1\}$ can only depend (at most) on $\{X_1, \ldots, X_{n-1}\}$ (Recall Section 1.3.) Since the sequence is assumed i.i.d., X_n is independent of $\{X_1, \ldots, X_{n-1}\}$ so that X_n is independent of the event $\{\tau > n-1\}$ yielding $E\{X_n I\{\tau > n-1\}\} = E(X)P(\tau > n-1)$. Taking the expected value of the above infinite sum thus yields (after bringing the expectation inside the sum; that's allowed here since $E(\tau)$ and $E|X|$ are assumed finite allowing use of Fubini's Theorem; see Remark 3 below.)

$$\begin{aligned} E\{\sum_{n=1}^{\tau} X_n\} &= E(X)\sum_{n=1}^{\infty} P(\tau > n-1) \\ &= E(X)\sum_{n=0}^{\infty} P(\tau > n) \\ &= E(X)E(\tau), \end{aligned}$$

where the last equality is due to "integrating the tail" method for computing expected values of non-negative r.v.s. ∎

Remark 2 Note that in the special case when τ is independent of the process $\{X_n : n \geq 1\}$, then a simple proof via conditioning on $\{\tau = k\}$ is possible:

$$\begin{aligned} E\{\sum_{n=1}^{\tau} X_n \mid \tau = k\} &= E\{\sum_{n=1}^{k} X_n \mid \tau = k\} \\ &= E\{\sum_{n=1}^{k} X_n\}, \text{via independence of } \{\tau = k\} \text{ and } \{X_n : n \geq 1\} \\ &= kE(X), \end{aligned}$$

and thus summing up over all $k \geq 1$ while unconditioning yields

$$\begin{aligned} E\{\sum_{n=1}^{\tau} X_n\} &= E(X)\sum_{k=1}^{\infty} kP(\tau = k) \\ &= E(X)E(\tau). \end{aligned}$$

For a general stopping time, the event $\{\tau = k\}$ is not independent of $\{X_n : n \geq 1\}$ which is why we can't use the above proof.

Remark 3 Interestingly, the way one uses Fubini's Theorem in the Proof of Wald's Equation involves first proving it when the X_n are non-negative and using Tonelli's theorem: We must show that taking the expected value the right hand side of Equation(1) is allowed to be interchanged with the sum. Fubini's Theorem says that if

$$E\sum_{n=1}^{\infty} |X_n| I\{\tau > n-1\} < \infty,$$

then the interchange is allowed. Tonelli's theorem is a special case of Fubini's Theorem which says that if the functions are non-negative, then the interchange is always allowed (finite or infinite). Thus we can use Tonelli's theorem to obtain

$$E\sum_{n=1}^{\infty} |X_n| I\{\tau > n-1\} = \sum_{n=1}^{\infty} E(|X_n|) I\{\tau > n-1\} = E(\tau)E(|X|).$$

The above is finite as long as both $E(\tau)$ and $E(|X|)$ are finite (this then allows us to use Fubini's Theorem for our original problem).

1.5 Applications of Wald's equation

1. Consider an i.i.d. sequence $\{X_n\}$ with a discrete distribution that is uniform over the integers $\{1, 2, \ldots, 10\}$; $P(X = i) = 1/10,\ 1 \le i \le 10$. Thus $E(X) = (1 + 10)/2 = 5.5$. Imagine that these are bonuses (in units of $\$10,000$) that are given to you by your employer each year. Let $\tau = \min\{n \ge 1 : X_n = 10\}$, the first time that you receive a bonus of size 10.

   What is the expected total (cumulative) amount of bonus received up to time τ?

   $$E\{\sum_{n=1}^{\tau} X_n\} = E(\tau)E(X) = 5.5E(\tau),$$

   from Wald's equation, if we can show that τ is a stopping time with finite mean.

   That τ is a stopping time follows since it is a first passage time: $\{\tau = 1\} = \{X_1 = 10\}$ and in general $\{\tau = n\} = \{X_1 \ne 10, \ldots, X_{n-1} \ne 10, X_n = 10\}$ only depends on $\{X_1, \ldots, X_n\}$.

   We need to calculate $E(\tau)$. Noting that $P(\tau = 1) = P(X_1 = 10) = 0.1$ and in general, from the i.i.d. assumption placed on $\{X_n\}$,

   $$\begin{aligned} P(\tau = n) &= P(X_1 \ne 10, \ldots, X_{n-1} \ne 10, X_n = 10) \\ &= P(X_1 \ne 10) \cdots P(X_{n-1} \ne 10)P(X_n = 10) \\ &= (0.9)^{n-1}0.1,\ n \ge 1, \end{aligned}$$

   we conclude that τ has a geometric distribution with "success" probability $p = 0.1$, and hence $E(\tau) = 1/p = 10$. And our final answer is $E(\tau)E(X) = 55$.

   Note here that If $\tau = n$, then before time n the random variables $X_1, \ldots, X_{n-1}$ no longer have the original uniform distribution; they are biased in that none of them takes on the value 10. So in fact they each have the conditional distribution $(X|X \ne 10)$ and thus an expected value different from 5.5. Moreover, the random variable at time $\tau = n$ has value 10; $X_n = 10$ and hence is not random at all. The point here is that even though all these random variables are biased, in the end, on average, Wald's equation let's us treat the sum as if they are not biased and are independent of τ as in Example 1 above.

   To see how interesting this is, note further that we would get the same answer 55 by using any of the stopping times $\tau = \min\{n \ge 1 : X_n = i\}$ for any $1 \le i \le 10$; nothing special about $i = 10$.

   This should indicate to you why Wald's equation is so important and useful.

   In this special example, we could in fact compute the answer by brute force: the conditional distribution $(X|X \ne 10)$ is the uniform distribution over the remaining 9 integers $\{1, 2, \ldots, 9\}$ (can you prove that?); thus $E(X|X \ne 10) = (1 + 9)/2 = 5$ and so

   $$E\{\sum_{n=1}^{\tau} X_n\} = E(\tau - 1)E(X|X \ne 10) + E(X_\tau) = (9)(5) + 10 = 55.$$

2. (*Null recurrence of the simple symmetric random walk*)

   $R_n = \Delta_1 + \cdots + \Delta_n,\ X_0 = 0$ where $\{\Delta_n : n \ge 1\}$ is i.i.d. with $P(\Delta = \pm 1) = 0.5$, $E(\Delta) = 0$. We already know that this MC is recurrent (proved via the gambler's ruin problem). That is, we know that the random time $\tau_{0,0} = \min\{n \ge 1 : R_n = 0|R_0 = 0\}$ is *proper*, e.g., is finite wp1. ($f_0 = 1$, where $f_0 = P(\tau_{0,0} < \infty)$.) But now we show that the chain is null recurrent, that is, that $E(\tau_{0,0}) = \infty$. *The chain will with certainty return back to state* 0*, but the expected number of steps required is infinite.*

We do so by proving that $E(\tau_{1,1}) = \infty$, where more generally, we define the stopping times $\tau_{i,j} = \min\{n \geq 1 : R_n = j | R_0 = i\}$. (Since the chain is irreducible, all states are null recurrent together or positive recurrent together; so if $E(\tau_{1,1}) = \infty$ then in fact $E(\tau_{j,j}) = \infty$ for all j.) In fact by symmetry, $\tau_{j,j}$ has the same distribution (hence mean) for all j.

By conditioning on the first step $\Delta_1 = \pm 1$,

$$\begin{aligned} E(\tau_{1,1}) &= (1 + E(\tau_{2,1}))1/2 + (1 + E(\tau_{0,1}))1/2 \\ &= 1 + 0.5E(\tau_{2,1}) + 0.5E(\tau_{0,1}). \end{aligned}$$

We will show that $E(\tau_{0,1}) = \infty$, thus proving that $E(\tau_{1,1}) = \infty$.

Note that by definition, the chain at the stopping time $\tau = \tau_{0,1}$ has value 1; for $\tau = \min\{n \geq 1 : R_n = 1 \mid R_0 = 0\}$,

$$1 = R_\tau = \sum_{n=1}^{\tau} \Delta_n.$$

Now we use Wald's equation with τ: for if in fact $E(\tau) < \infty$ then we conclude since $E(\Delta) = 0$ that

$$1 = E(R_\tau) = E\Big\{\sum_{n=1}^{\tau} \Delta_n\Big\} = E(\tau)E(\Delta) = 0,$$

yielding the contradiction $1 = 0$; thus $E(\tau) = \infty$.

1.6 Strong Markov property

Consider a Markov chain $\mathbf{X} = \{X_n : n \geq 0\}$ with transition matrix P. The Markov property can be stated as saying: *Given the state X_n at any time n (the present time), the future $\{X_{n+1}, X_{n+2}, \dots\}$ is independent of the past $\{X_0, \dots X_{n-1}\}$.*

If τ is a stopping time with respect to the Markov chain, then in fact, we get what is called the *Strong Markov Property*: *Given the state X_τ at time τ (the present), the future $\{X_{\tau+1}, X_{\tau+2}, \dots\}$ is independent of the past $\{X_0, \dots X_{\tau-1}\}$.*

The point is that we can replace a deterministic time n by a stopping time τ and retain the Markov property. It is a stronger statement than the Markov property.

This property easily follows since $\{\tau = n\}$ only depends on $\{X_0, \dots, X_n\}$, the past and the present, and not on any of the future: Given the joint event $\{\tau = n, X_n = i\}$, the future $\{X_{n+1}, X_{n+2}, \dots\}$ is still independent of the past:

$$\begin{aligned} P(X_{n+1} = j \mid \tau = n, X_n = i, \dots, X_0 = i_0) &= P(X_{n+1} = j \mid X_n = i, \dots, X_0 = i_0) \\ &= P(X_{n+1} = j \mid X_n = i) \text{ via the Markov Property} \\ &= P_{i,j}. \end{aligned}$$

This being true regardless of the value of n, we have just proved:

Theorem 1.2 *Any discrete-time Markov chain satisfies the Strong Markov Property.*

We actually have used this result already without saying so: Every time the rat in the maze returned to cell 1 we said that "The chain starts over again and is independent of the past". Formally, we were using the strong Markov property with the stopping time

$$\tau = \tau_{1,1} = \min\{n \geq 1 : X_n = 1 | X_0 = 1\}.$$

Corollary 1.1 *As a consquence of the strong Markov property, we conclude that the chain from time* τ *onwards,* $\{X_{\tau+n} : \ n \geq 0\}$*, is itself the same Markov chain but with initial condition* $X_0 = X_\tau$*.*

For example in the context of the rat in the maze with $X_0 = 1$, we know that whenever the rat enters cell 2, the rats movement from then onwards is still the same Markov chain but with the different initial condition $X_0 = 2$.