# 1 Expected number of visits of a finite state Markov chain to a transient state

When a Markov chain is not positive recurrent, hence does not have a limiting stationary distribution $\pi$, there are still other very important and interesting things one may wish to consider computing. For example, in the rat in the open maze, we computed the expected number of moves until the rat escapes. In the Gambler's Ruin Problem, we considered the probability that the game ends in no more than (say) 7 gambles, and so on.

Here we consider another such important quantity for finite state chains that have transient states: The expected number of visits of the chain to a transient state.

Consider a finite state space (N=$|\mathcal{S}| < \infty$) MC with $1 \le b < N$ transient states, labeled $1, \ldots, b$. (Without loss of generality we can always assume the $b$ states are ordered this way via relabeling the states.) Let $T = \{1, \ldots, b\}$, denote the set of transient states and let $P_T = (P_{i,j}), i, j \in T$, denote the transition matrix for only these transient states, a $b \times b$ matrix. (**But not a stochastic matrix; rows need not sum to 1 now!**)

Recall that a state $i$ is called transient if $f_i < 1$, where $f_i$ denotes the probability the chain will ever return to state $i$ in the future given that $X_0 = i$. Thus such a state is visited only a finite (but random) number of times and then never again. Letting $N_i$ denote the total number of times that a transient state $i$ is visited given that $X_0 = i$, we know that $N_i$ has a geometric distribution:

$$P(N_i = k) = f_i^{k-1}(1 - f_i),\ k \ge 1,$$

where we are counting the initial visit from $X_0 = i$ as the first visit. Thus $E(N_i) = 1/(1 - f_i)$ is the expected number of visits to transient state $i$ given that $X_0 = i$. Of course, we would need to know the value $f_i$ in order to explicitly compute $E(N_i)$. We will learn how to do that and more so in what follows.

(The other states are called recurrent; $f_i = 1$; they are visited over and over, always returned to again, an infinite number of times.)

Let $S_{i,j}$ = the expected number of times (over all time) that the chain visits state $j \in T$ given $X_0 = i \in T$.

$$S_{i,j} = E\{\sum_{n=0}^{\infty} I\{X_n = j | X_0 = i\}\} = \sum_{n=0}^{\infty} P_{ij}^n,$$

$S = (S_{i,j})$ is a $b \times b$ matrix [1]. Note that $S_{i,i} \ge 1$ because the initial visit $X_0 = i$ is counted, and $S_{i,i} = E(N_i)$ as discussed above.

**Proposition 1.1** *Let $I$ denote the $b \times b$ identity matrix. Then $S = I + P_T S$ yielding the solution*

$$S = (I - P_T)^{-1}. \tag{1}$$

[1] Observe that it is not possible for the chain to make a transition from a recurrent state to a transient state for otherwise (since the recurrent state is visited infinitely often), the transient state would be visited infinitely often too contradicting that it is transient. Thus, once the chain makes a transition from a transient state to a recurrent state it never returns back to the set $T$. As a result, we only must consider transitions from states in $T$ to states in $T$; $P_T$.

*Proof :* Suppose that $X_0 = i \in T$. We first consider the case $j = i$. Since $X_0 = i$ we already have one such (the initial) visit to $i$. Thus

$$S_{i,i} = 1 + \sum_{k \in T} P_{i,k} S_{k,i},$$

by conditioning on the first state visited, $X_1 = k \in \mathcal{S}$, where we are using the Markov property as soon as the chain visits the state $k$ to add on the additional visits to $i$ starting initially in state $k$ (e.g., $S_{k,i}$).

For the case $j \neq i$ we do not have any initial visit to state $j$, hence

$$S_{i,j} = \sum_{k \in T} P_{i,k} S_{k,j}.$$

We combine the two cases in matrix form and obtain the result $S = I + P_T S$, yielding $(I - P_T)S = I$. But since in general, for any two square matrices, $A$, $B$ it holds that $det(AB) = det(A)det(B)$ (multiplicative property of determinants), we conclude (via letting $A = I - P_T$ and $B = S$; $det(AB) = det(I) = 1$) that both $S$ and $I - P_T$ have non-zero determinant, hence are invertible; we thus can write the solution $S = (I - P_T)^{-1}$.

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As an example, consider the rat in the open maze; for then $T = \{1, 2, 3, 4\}$, 0 is not included since it is recurrent, and

$$P_T = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix}.$$

Thus,

$$I - P_T = \begin{pmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/2 & 1 & 0 & -1/2 \\ -1/2 & 0 & 1 & -1/2 \\ 0 & -1/3 & -1/3 & 1 \end{pmatrix}.$$

Thus,

$$S = (I - P_T)^{-1} = \begin{pmatrix} 4 & 3 & 3 & 3 \\ 3 & 3.5 & 2.5 & 3 \\ 3 & 2.5 & 3.5 & 3 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

$S_{1,4} = 3$, for example, is the expected number of times that the rat visits room 4 before escaping, given that it initially started off in room 1.

As another interesting example, we can consider the Gambler's Ruin Problem, with state space $\mathcal{S} = \{0, 1, 2, , \ldots, N\}$; $T = \{1, 2, \ldots, N - 1\}$, states 0 and $N$ are not included since they are recurrent. By computing $S = (I - P_T)^{-1}$, in the case when (say) $N = 10$, we could, for example find the expected number of gambles (before the game ends) that the gambler's total is exactly \$6 given that this gambler starts with \$2; $S_{2,6}$.

**Remark 1** Note that we can use the matrix $S$ to re-compute the expected number of moves until the rate escapes in the open maze problem: For example, $S_{1,1} + S_{1,2} + S_{1,3} + S_{1,4} = 13 = E(T_{1,0})$ = the expected number of moves until the rat escapes, given the rat starts in Room 1.

## 1.1 Probability of ever visiting state $j$ given the chain is initially in state $i \neq j$. state

For transient states $i, j \in T$, with $i \neq j$, let $f_{i,j}$ denote the probability that the Markov chain ever visits state $j$ given $X_0 = i$. (For example, the probability that the rat, starting in Room 2 will ever visit Room 3 before escaping.) It is immediate that

$$S_{i,j} = f_{i,j} S_{j,j} + (1 - f_{i,j})0 = f_{i,j} S_{j,j},$$

because if the chain never visits $j$ (probability $1 - f_{i,j}$), then the total number of visits to $j$ is zero, whereas if the chain does visit $j$ (probability $f_{i,j}$), then by the Markov property, the chain is as if initially in state $j$ and we want $S_{j,j}$ (which includes the first visit by definition).

We thus derive

$$f_{i,j} = \frac{S_{i,j}}{S_{j,j}}. \tag{2}$$

For the rat in the open maze, we know that $f_{i,4} = 1$ for $i = 1, 2, 3$ because the rat must visit Room 4 inorder to escape, but the other $f_{i,j}$ are not 1. For example $f_{2,3} = S_{2,3}/S_{3,3} = 2.5/3.5 = 5/7 < 1$; the rat might never visit Room 3; could for example go $2 \to 4 \to 0$.