IEOR 4106
lec 1

Intro to Markov Chains

via the Simple Random Walk, Gambler's Ruin Problem

A stochastic process (discrete time) is a sequence of random variables (rvs)

$X_0, X_1, X_2, \ldots$

$= \{X_n : n \geq 0\}$ $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$

X_n = state of the process "at time n"

A very special case:
when the rvs $\{X_n\}$ are independent and identically distributed (iid)

(they all have the <u>same</u> distribution)

<u>CDF</u> $F(x) = \mathbb{P}(X_n \leq x)$, $x \in \mathbb{R}$

for all $n \in \mathbb{N}$

Strong Law of Large Numbers (SLLN):

If $\{X_n : n \geq 1\}$ is iid and $E|X| < \infty$ then

$$\mathbb{P}\left(\lim_{n\to\infty} \boxed{\frac{1}{n}\sum_{i=1}^{n} X_i} = E(X)\right) = 1$$

(w.p. 1)

the "state space" $\mathcal{S}$ of a stochastic process $\{X_n\}$ is the collection of all values that the X_n take on

$\mathcal{S}$ examples:

$\mathcal{S} = \mathbb{R} = (-\infty, \infty)$

$\mathcal{S} = \mathbb{N}$, $\mathcal{S} = \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

or a finite subset of $\mathbb{Z}$

$$\mathcal{S} = \mathbb{R}^+ = [0, \infty)$$

$$\left(\mathcal{S} = \mathbb{R}^d\right)$$

$$d \geq 2$$

fair (a) coin flipping i.i.d example

generalizes to i.i.d "Bernoulli(p)" trials (X_n)

$E(X) = 1 \cdot p + 0 \cdot (1-p) = p$

$\mathbb{P}(X=1) = p$ = prob of "success"

$\mathbb{P}(X=0) = 1-p$ = prob. of "failure"

$\left(\begin{array}{c} 0 < p < 1 \\ \text{fixed} \end{array}\right)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = p \quad \text{wp1}$$

via SLLN

$\boxed{\text{Simple}}$ Random Walk

Start with $\underline{\{\Delta_n : n \geq 1\}}$ iid

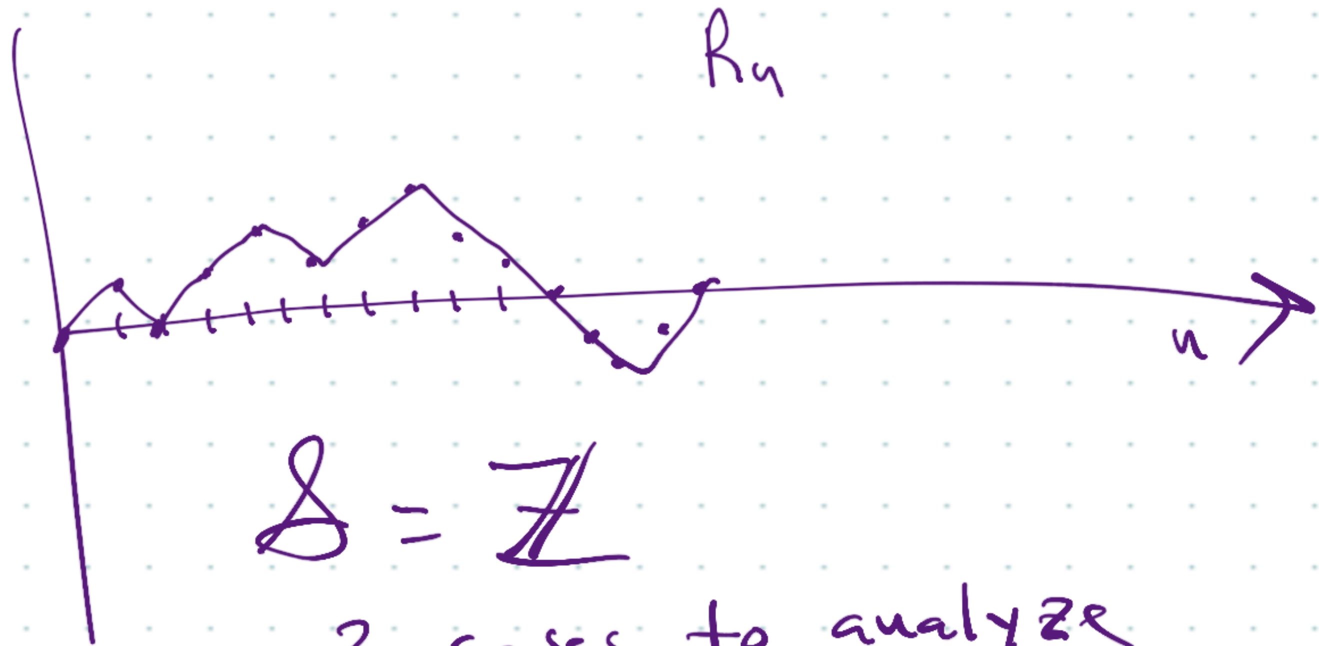
$\mathbb{P}(\Delta = 1) = p$

$\mathbb{P}(\Delta = -1) = 1-p = q$

$R_0 = 0$ $\quad R_n = \Delta_1 + \cdots + \Delta_n, \quad n \geq 1$

"Position after the n^{th} step"

$\boxed{\{R_n : n \geq 0\} \text{ is called a Random walk}}$



$$S = \mathbb{Z}$$

3 cases to analyze

1) $E(\Delta) = 1 \cdot p - 1(1-p) = 2p-1$
> 0 (positive drift)

2) $E(\Delta) < 0$ (negative drift)

3) $E(\Delta) = 0$ $\left(p = \frac{1}{2}\right)$ Symmetric case

If case (1) ($E(\Delta) > 0$)

then $\lim_{n \to \infty} R_n = +\infty$ wp1

$$\mathbb{P}\left(\lim_{n \to \infty} R_n = +\infty\right) = 1$$

<u>Proof:</u> Let $\alpha = E(\Delta) > 0$

SLLN: $\frac{1}{n} \sum_{i=1}^{n} \Delta_i \xrightarrow[n \to \infty]{} E(\Delta) = \alpha > 0$ wp1

$= \frac{R_n}{n}$

$\Rightarrow$ $R_n \approx \underline{n\alpha}$ as $n \to \infty$

$\longrightarrow +\infty$

Case (2) Same proof

$$R_n \xrightarrow[n\to\infty]{} -\infty \quad \text{wp1}$$

Case 3

Proved later

$$\left(\begin{array}{l} \limsup_{n\to\infty} R_n = +\infty \quad \text{wp1} \\ \liminf_{n\to\infty} R_n = -\infty \quad \text{wp1} \end{array} \right)$$

More generally we can define random walks that are not simple:

$\{\Delta_n\}$ are iid

with $F(x) = \mathbb{P}(\Delta \le x)$, $x \in \mathbb{R}$

(could be a Normal dist. for example $N(\mu, \sigma^2)$)

$$R_n = \sum_{i=1}^{n} \Delta_i, \quad R_0 = 0$$

Recursive nature of random walks:

$$\boxed{R_{n+1} = R_n + \Delta_{n+1} \quad n \geq 0}$$

Let's focus on the Simple case

and study a

famous problem

Gambler's Ruin Problem

$\{\Delta_n\}$ iid $\mathbb{P}(\Delta = 1) = p$
$\mathbb{P}(\Delta = -1) = q = 1-p$

$\Delta_n =$ amount earned by a gambler from n^{th} gamble

$X_n =$ total earnings after n^{th} gamble, $n \geq 1$

$i = X_0 =$ initial \$ ≥ 1

Fix $N \geq 2$

$X_0 = i$, $1 \leq i \leq N-1$ initial fortune

$$\mathbb{P}(X_n \text{ hits } N \text{ before hitting } 0 \mid X_0 = i)$$

$$= P_i(N) = ?$$

$$1 - P_i(N) = \mathbb{P}(X_n \text{ hits } 0 \text{ before hitting } N \mid X_0 = i)$$

"Gambler is ruined"

$$\begin{pmatrix} P_0(N) = 0 \\ P_N(N) = 1 \end{pmatrix} \qquad P_i = P_i(N)$$

condition on 1rst gamble

$(\Delta_i = \pm 1)$

$$\boxed{P_i = \boxed{p\, P_{i+1}} + q \boxed{P_{i-1}}}$$

$(P_0 = 0,\ P_N = 1)$

In general
for any rv X,

$$E(X) = E(X \mid \Delta_i = 1)\mathbb{P}(\Delta_i = 1) + E(X \mid \Delta_i = -1)\mathbb{P}(\Delta_i = -1)$$

$$= pE(X \mid \Delta_i = 1) + (1-p)E(X \mid \Delta_i = -1)$$

That leads to $\boxed{P_i = p P_{i+1} + (1-p) P_{i-1} \quad 1 \le i \le N-1}$

$(P_0 = 0,\ P_N = 1)$

Solution can be derived:
(using basic algebra) (See lecture Notes for the derivation)

$$P_i(N) = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & p \neq q \quad (p \neq \tfrac{1}{2}) \\[2ex] i/N & p = q = \tfrac{1}{2} \end{cases}$$

$$P_i(\infty) = \lim_{N\to\infty} P_i(N) = \begin{cases} 1-(q/p)^i & p > \tfrac{1}{2} \\ 0 & p \leq \tfrac{1}{2} \end{cases}$$

$$P_i(\infty) = \mathbb{P}\left(\text{Gambler (if allowed to play forever) becomes infinitely Rich without ever going broke} \;\middle|\; X_0 = i\right)$$

$$1 - P_i(\infty) = \mathbb{P}\left(\text{Gambler if allowed to play forever goes broke} \;\middle|\; X_0 = i\right)$$