IEOR 4106 lec 11

More on the Poisson process:

Partitioning Poisson rvs and poisson processes

A renewal (point) process is

a sequence of points

$0 < t_1 < t_2 < \cdots$

$\lim_{n \to \infty} t_n = \infty$

such that

the interarrival times $X_n = t_n - t_{n-1}$, $n \geq 1$

are iid $\quad (t_0 \stackrel{\text{def}}{=} 0)$

$F(x) = \mathbb{P}(X \leq x)$, $x \geq 0 \quad \lambda \stackrel{\text{def}}{=} \dfrac{1}{E(X)}$

Elementary Renewal Theorem
(ERT)

for the counting process $\{N(t) \,;\, t \geq 0\}$

$N(t) \overset{def}{=}$ the number of points (arrivals) that occur during $(0, t]$

$$\boxed{\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \left(= \frac{1}{E(X)}\right) \quad wp\,1}$$

$$\left(\lim_{t\to\infty} \frac{E(N(t))}{t} = \lambda\right)$$

a Poisson process is $\boxed{PP(\lambda)}$

a renewal process with

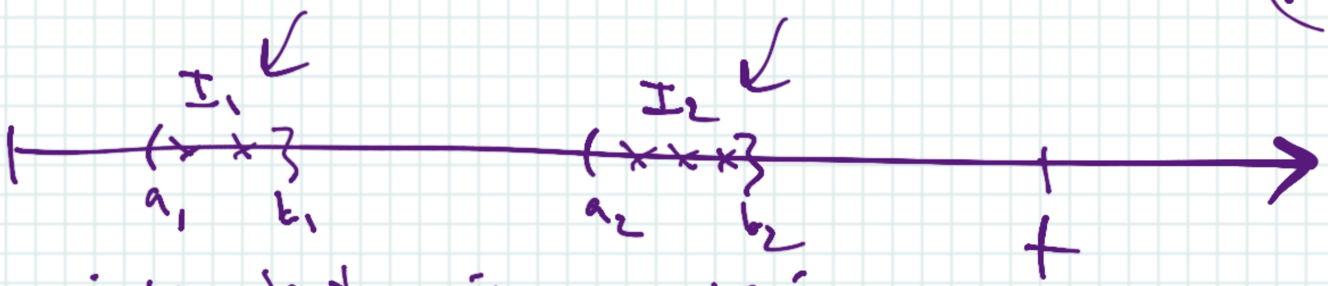
$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0,$$ the exponential distribution at rate λ.

For a $PP(\lambda)$ $(N(t) : t \geq 0)$ has both stationary and independent increments; the increments have a Poisson dist.:

$$\mathbb{P}(N(s+t) - N(s) = k) = \mathbb{P}(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k \geq 0$$

$E(N(t)) = \lambda t$, $t \geq 0$ Poisson dist, mean λt

$(N(0) \stackrel{\text{def}}{=} 0)$

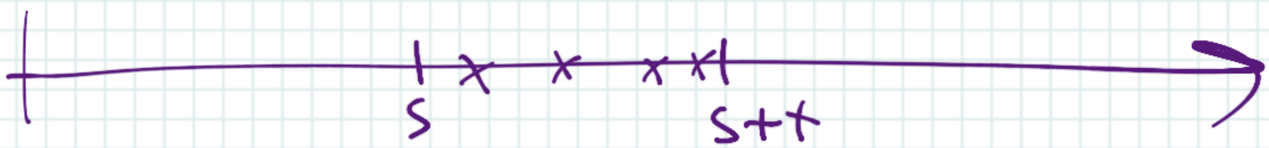


<u>independent</u> increments:

$N(I_1)$ and $N(I_2)$ are independent r.v.s.

Stationary increments: for each fixed $t \geq 0$,

$N(s+t) - N(s)$ has the <u>same</u> distribution for all $s \geq 0$; same as $N(t)$.



Fact: If, for a counting process $(N(t) : t \geq 0)$

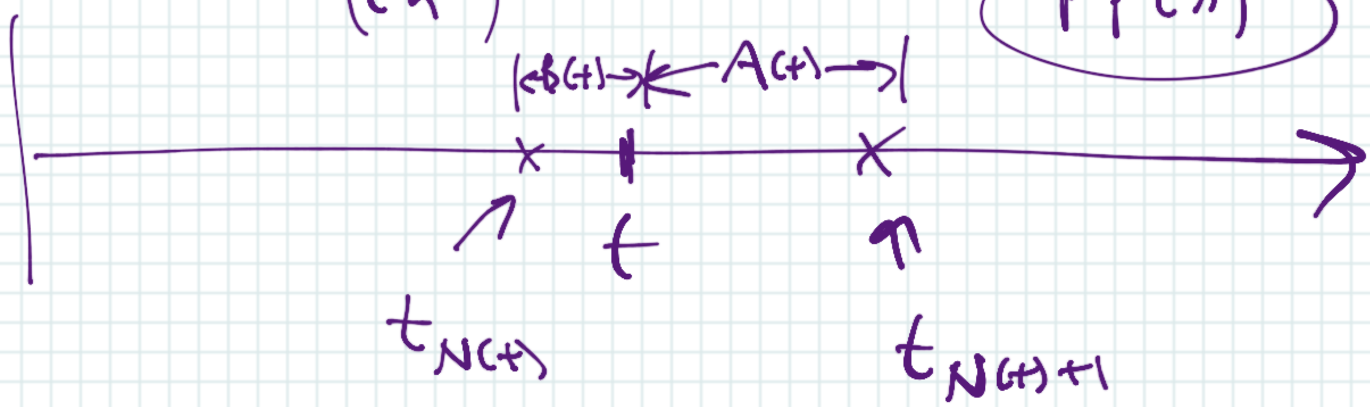
it has both stationary & independent increments

then the point process is a $PP(\lambda)$ (some $\lambda > 0$)

Most point processes $\{t_n ; n \geq 1\}$
do not have either property

Suppose, for example we have a renewal process
with $F(x) = \mathbb{P}(X \leq x) = x, \quad x \in (0,1)$
unif(0,1) distr

$\{t_n\}$ $\qquad$ PP(λ)



$$X_{N(t)+1} = t_{N(t)+1} - t_{N(t)}$$

length of the interarrival time covering t

$$A(t) = t_{N(t)+1} - t$$
$$B(t) = t - t_{N(t)}$$

$A(t) \sim \exp(\lambda)$
and is independent of $B(t)$
memoryless property

$X_{N(t)+1} = B(t) + A(t)$

is larger than just

a typical X_n (iid $\exp(\lambda)$)

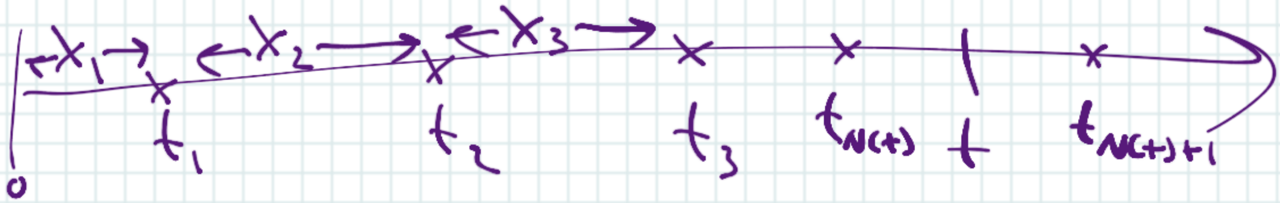
Suppose

$\{X_n\} = \{1, 2, 1, 3, \cdots\cdots\}$

X_n = lifetime of n^{th} lightbulb

$A(t)$ = remaining lifetime of the bulb you find burning at time t

$B(t)$ = age of the bulb



Partitioning of Poisson rvs, and Poisson processes

(1) Suppose $X \sim \text{Poisson}(\alpha)$

$$\mathbb{P}(X=k) = e^{-\alpha}\frac{\alpha^k}{k!}, \quad k \geq 0$$

$$X \xrightarrow{p} X_1 = \sum_{i=1}^{X} I\{B_i = 1\}$$

$$X \xrightarrow{q = 1-p} X_2 = \sum_{i=1}^{X} I\{B_i = 0\}$$

Let $B_1, B_2, \dots$ be iid Bernoulli(p) rvs

$$\boxed{X = X_1 + X_2}$$

Then $X_1 \sim \text{Poisson}(p\lambda)$

$X_2 \sim \text{Poisson}(q\lambda)$ $\quad q = 1-p$

and they are independent rvs !

<u>proof:</u>

must show

$$\mathbb{P}(X_1 = k, X_2 = \ell) = \boxed{e^{-\lambda p}\frac{(\lambda p)^k}{k!} \times e^{-\lambda q}\frac{(\lambda q)^\ell}{\ell!}}$$

$\quad$ ‖

$\mathbb{P}(X_1 = k, \underline{X = k+\ell})$ $\qquad k \geq 0, \ell \geq 0$

$$= \mathbb{P}(X_1 = k \mid X = k+\ell)\, \mathbb{P}(X = k+\ell) \qquad \text{(algebra)}$$

$(X = X_1 + X_2)$ $\qquad \binom{k+\ell}{k} p^k q^\ell \qquad e^{-\lambda}\frac{\lambda^{k+\ell}}{(k+\ell)!}$ $\qquad \lambda = p\lambda + q\lambda$ $(p+q=1)$

Partitions of a $PP(\lambda)$

$$\Psi = \{t_n : n \geq 0\}$$

(2)

$$\Psi \xrightarrow{p} \Psi_1 = \{t_n(1) : n \geq 1\} \sim PP(\lambda_1) \quad \lambda_1 = p\lambda$$

$$\Psi \xrightarrow[q = 1-p]{} \Psi_2 = \{t_n(2) : n \geq 1\} \sim PP(\lambda_2) \quad \lambda_2 = q\lambda$$

and Ψ_1, Ψ_2 are independent

(2) can be shown to follow from (1)

$(N_1(t))$

$(N_2(t))$

$$E(N_1(10) \mid N_2(10) = 10^6)$$

$$= E(N_1(10)) = \lambda_1(10) = p\lambda(10)$$

Consider $t_n = n,\ n \geq 1$, deterministic Renewal Process

$| N_2(10) = 8$

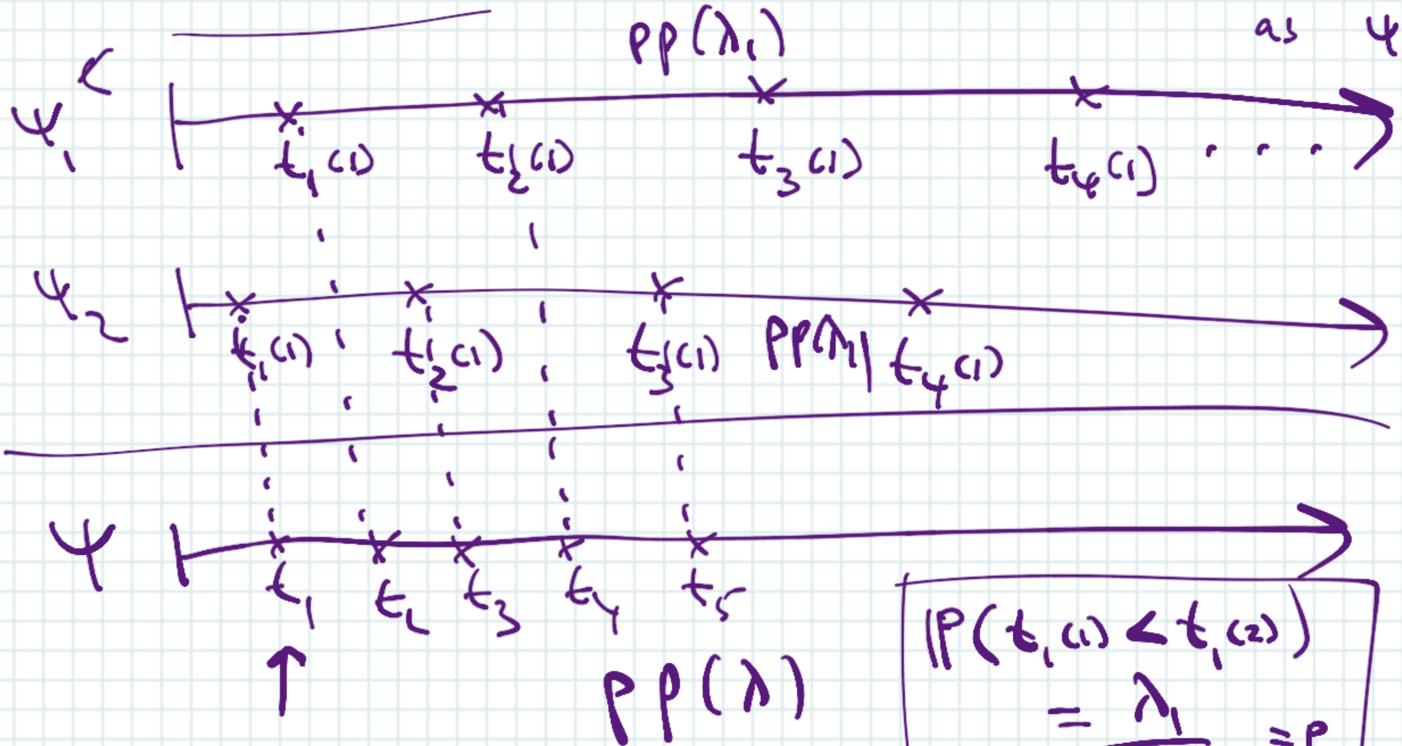
In general, does not hold for renewal processes

always, $(N_1(n) = n - N_2(n)) \Rightarrow N_1(10) = 2$

$\Psi \xrightarrow{p} \Psi_1$

$\Psi \xrightarrow{q} \Psi_2$

Superposition: putting Ψ_1, Ψ_2 back together as Ψ



$$\mathbb{P}(t_1(1) < t_1(2)) = \frac{\lambda_1}{\lambda_1+\lambda_2} = p$$

Let ψ denote the superposition of two independent PP

$\boxed{\psi_1 \sim PP(\lambda_1)}$

$\boxed{\psi_2 \sim PP(\lambda_2)}$

Then $\psi \sim PP(\lambda)$ $\quad \lambda = \lambda_1 + \lambda_2$

$$\boxed{\begin{array}{l} \psi \xrightarrow{p} \psi_1 \\ \psi \xrightarrow{q} \psi_2 \end{array}}$$

$$\boxed{p = \frac{\lambda_1}{\lambda_1+\lambda_2}} \quad \boxed{q = \frac{\lambda_2}{\lambda_1+\lambda_2}}$$

Order Statistics of n iid rvs

$t > 0$

Suppose $V_1, V_2, \dots V_n$ are iid $\quad n \geq 1$

Unif$(0, t)$

$$\mathbb{P}(V \leq s) = \frac{s}{t}, \quad 0 < s < t$$

density $h(s) = \frac{1}{t}, \quad 0 < s < t.$

Put them in ascending order

$$\boxed{V_{(1)} < V_{(2)} < \dots < V_{(n)}}$$

Joint density of $(V_1, V_2, \dots, V_n)$

$$g(s_1, s_2, \dots, s_n) = \boxed{\frac{1}{t^n}} = h(s_1)h(s_2)\cdots h(s_n)$$

$s_i \in (0, t)$
$1 \le i \le n$

(product of n iid)

$\Rightarrow$ Joint density of $(V_{(1)}, V_{(2)}, \dots, V_{(n)})$

$$h(s_1, s_2 \cdots s_n) = \frac{n!}{t^n}$$

$0 < s_1 < s_2 < \cdots < s_n < t$

(There are $n!$
$= n(n-1)(n-2)\cdots 1$
ways to
permute the $V_1, V_2 \cdots V_n$
to obtain the identical
ordering $(V_{(1)}, \cdots V_{(n)})$)

Suppose $t=1$

$V_1 = .4, \; V_2 = .72$

$(V_{(1)}, V_{(2)}) = (.4, .72)$

$V_1 = .72, \; V_2 = .4$

$(V_{(1)}, V_{(2)}) = \boxed{(.4, .72)}$ for both

$V_1 = .4, \; V_2 = .72, \; V_3 = .89$

$(V_{(1)}, V_{(2)}, V_{(3)}) = \boxed{(.4, .72, .89)}$

$V_1 = .72, \; V_2 = .4, \; V_3 = .89$

$V_1 = .89, \; V_2 = .72, \; V_3 = .4$ etc.; $3! = 6$ possibilities

$\vdots$

Thm: for a $PP(\lambda)$ $(t>0)$

conditional on $\{N(t)=n\}$,

The joint distribution of

$\rightarrow (t_1, t_2, \ldots, t_n)$ $\quad 0<t_1<t_2<\cdots<t_n<t$

is that of the order statistics

$\boxed{(V_{(1)}, V_{(2)}, \ldots, V_{(n)})}$ of n

iid unif$(0,t)$ rvs $V_1, V_2, \ldots V_n$

$$\boxed{t_1 = V_{(1)},\ t_2 = V_{(2)},\ t_3 = V_{(3)} \cdots,\ t_n = V_{(n)}}$$

Joint density of $\left((t_1, t_2, \cdots, t_n) \,\middle|\, N(t) = n \right)$

is $\frac{n!}{t^n} = h(s_1, \cdots, s_n)$
$0 < s_1 < s_2 < \cdots < s_n < t$,

Same as the Joint density of

$(V_{(1)}, V_{(2)}, \cdots, V_{(n)})$ order statistics

of $V_1 \cdots V_n$ iid Unif$(0,t)$ rvs.