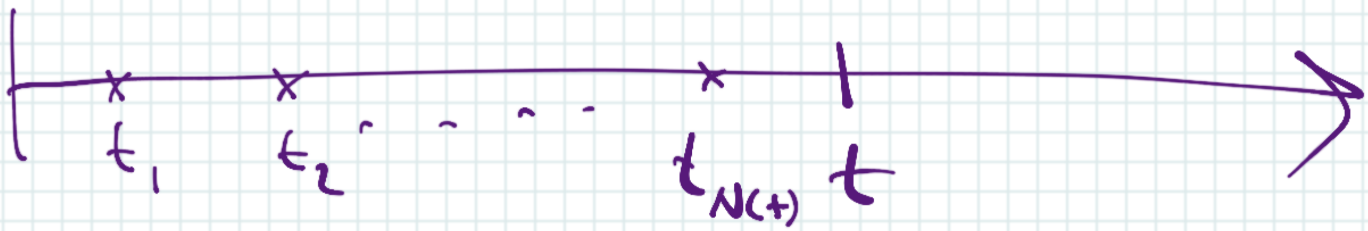
IEOR 4106 lec 12

Applications of order statistics

Construction of a Poisson process.

M|G|∞ Queue

Order Statistics representation for a Poisson process up to a fixed time $\underline{t > 0}$



Conditional on $\{N(t) = n\}$,

the joint distribution of the n arrival times $(t_1, t_2, \cdots, t_n)$ is the same as the order statistics of n iid Unif$(0,t)$ rvs $V_1, V_2, \cdots V_n$

$$(V_{(1)}, V_{(2)}, \cdots, V_{(n)}) \quad \left(t_i \stackrel{\text{def}}{=} V_{(i)} \atop 1 \le i \le n \right)$$

density $\left(\frac{n!}{t^n}\right)$

$n=1$ proof: (Fix $t>0$)

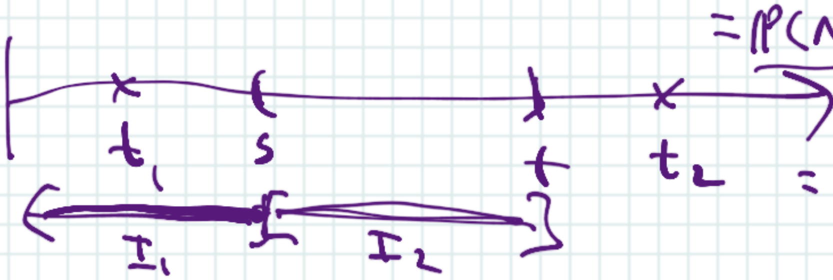
We must show that

$$(t_1 \mid N(t)=1) \sim \text{unif}(0,t]$$

i.e. $\boxed{\mathbb{P}(t_1 \le s \mid N(t)=1) = \frac{s}{t}}$, $\quad 0 \le s \le t$

$$= \frac{\mathbb{P}(t_1 \le s,\ N(t)=1)}{\mathbb{P}(N(t)=1)} = \frac{\mathbb{P}(N(s)=1,\ N(t)-N(s)=0)}{\mathbb{P}(N(t)=1)}$$

$$= \frac{\mathbb{P}(N(s)=1)\,\mathbb{P}(N(t-s)=0)}{\mathbb{P}(N(t)=1)}$$ (ind. & stat. increments property)



$$= \frac{e^{-\lambda s}\lambda s \ e^{-\lambda(t-s)}}{e^{-\lambda t}\ \lambda t} = \frac{s}{t}$$

$n=2$ $\qquad \mid N(t)=2$

$V_{(1)} = \min(V_1, V_2)$ $\qquad V_1, V_2$

$V_{(2)} = \max(V_1, V_2)$ $\qquad$ iid unif$(0,t)$

$$(t_1, t_2) \overset{dist}{=} (V_{(1)}, V_{(2)})$$

$$\mathbb{P}(V_{(1)} > s) = \mathbb{P}(\min(V_1, V_2) > s)$$

$0 \leq s \leq t$

$$= \mathbb{P}(V_1 > s, V_2 > s)$$

$$= \mathbb{P}(V_1 > s)\,\mathbb{P}(V_2 > s) = \mathbb{P}^2(V > s)$$

$$= \frac{(t-s)^2}{t^2}$$

$$\mathbb{P}(V_{(2)} \le s) = \mathbb{P}(\max(V_1, V_2) \le s)$$

$$= \mathbb{P}(V_1 \le s, V_2 \le s)$$

$$= \mathbb{P}(V_1 \le s)\,\mathbb{P}(V_2 < s) = \left(\frac{s}{t}\right)^2$$

$t = 1$ and $n = 2$

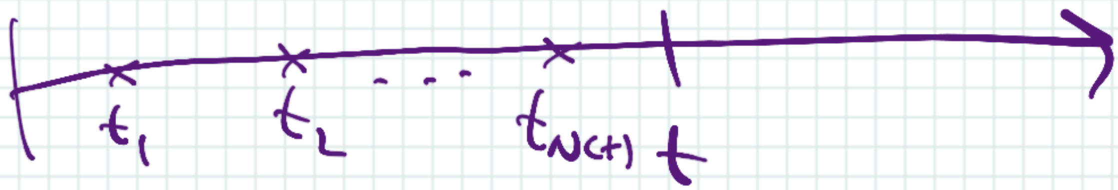
$\mathbb{P}(V_{(1)} > s) = (1-s)^2 \qquad 0 \le s \le t \qquad$ density $\boxed{2(1-s)}$

$\mathbb{P}(V_{(2)} > s) = 1 - s^2 \qquad \boxed{2s}$

$$\boxed{\begin{array}{l} f(x) = c\,x^n (1-x)^m \qquad n \ge 0,\ m \ge 0 \\ \text{density} \qquad 0 < x < 1 \end{array}}$$

beta distributions

Application: passengers arrive to a bus platform $\sim PP(\lambda)$, bus will leave (holding all passengers) at time t



waiting time of i^{th} passenger $\quad W_i = t - t_i$

What is expected average waiting time of a passenger?

$$\mid N(t) = n \qquad (n \geq 1)$$

$$(t_1, t_2, \cdots, t_n) = (V_{(1)}, V_{(2)}, \cdots, V_{(n)})$$

$$T = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} (t - t_i) = \sum_{i=1}^{n} (t - V_{(i)})$$

We want $\frac{E(T)}{n} = \frac{n\,E(t-V)}{n} = \boxed{\frac{t}{2}}$ $= \sum_{i=1}^{n} (t - V_i)$

If $V \sim \text{unif}[0,t]$,

then $t - V \sim \text{unif}[0,t]$

$$\mathbb{P}(t - V \leq s) = \mathbb{P}(V \geq t-s)$$

$$= \frac{t-(t-s)}{t} = \frac{s}{t} \checkmark$$

$$0 \leq s \leq t$$

$M|G|\infty$ Queueing model

$M|G|1$ Queue

$\{t_n\}$ is a $PP(\lambda)$

n^{th} customer arrives at time t_n

$$\boxed{\rho = \frac{\lambda}{\mu}}$$

$$\left(\begin{array}{c} 0<\lambda<\infty \\ 0<\mu<\infty \end{array}\right)$$

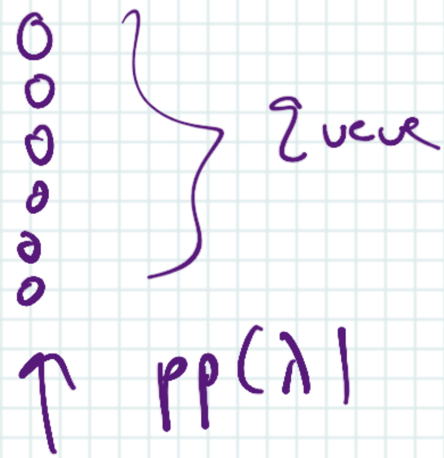
iid service times (S_n)

$G(x) = \mathbb{P}(S \le x),\ x \ge 0$

$PP(\lambda)$ $\{t_n\}$

$$\boxed{\frac{1}{\mu} \overset{\text{def}}{=} E(S)}$$

M|G|c (c servers in parallel



If $1 \leq c < \infty$, then there is a "queue" (line)

If $c = \infty$, no queue.

$M|G|\infty \quad (c = \infty)$

$\cdots \Box\Box\Box\Box - - - - - - -$ (S_n) iid $G(x)$

(No queue)

$\uparrow$ $PP(\lambda)$ $\{t_n\}$

$\rho = \lambda/\mu$ (No line) nth customer arrives at time t_n, departs at time $\underline{t_n + S_n = t_n^d}$

Let $L(t)$ = number of busy servers at time t

$(L(0) = 0)$

What is the distribution, for any fixed $t > 0$, of $L(t)$?

$$\mathbb{P}(L(t) = k) = \, ?$$

$k \geq 0$

Prop: for each $t > 0$,

$$\boxed{\frac{1}{\mu} \stackrel{\text{def}}{=} E(S)}$$

$L(t)$ has a Poisson distribution with mean $\alpha(t)$

$$\alpha(t) = \lambda \int_0^t \mathbb{P}(S > u)\,du = \lambda \int_0^t \bar{G}(u)\,du$$

$$\mathbb{P}(L(t) = k) = e^{-\alpha(t)} \frac{(\alpha(t))^k}{k!}, \quad k \geq 0 \qquad \left(\begin{array}{l} E(L(t)) \\ = \alpha(t) \end{array} \right)$$

---

observe $\lim_{t \to \infty} \alpha(t) = \lambda \int_0^\infty \bar{G}(u)\,du = \lambda E(S) = \rho$

$\Longrightarrow$

$$\mathbb{P}(L(t)=k) \xrightarrow[t\to\infty]{} e^{-\rho}\,\frac{\rho^k}{k!}\,, \quad k\geq 0.$$

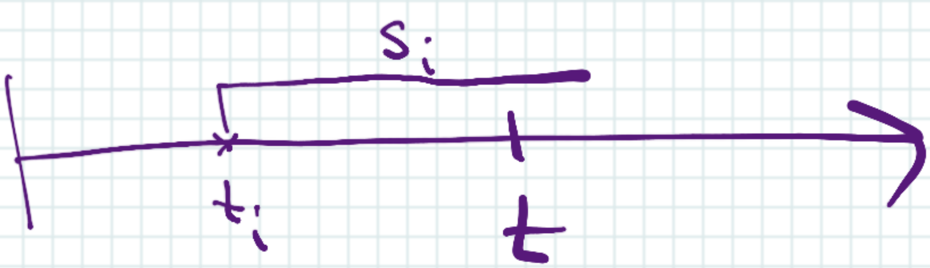
$L(t)$ converges in distribution (limiting dist. of $L(t)$) to the Poisson(ρ)

Proof: partition the Poisson rv $N(t)$

$$N(t) \xrightarrow{p(t)} L(t) = \boxed{\sum_{i=1}^{N(t)} I\{S_i > t - t_i\}}$$

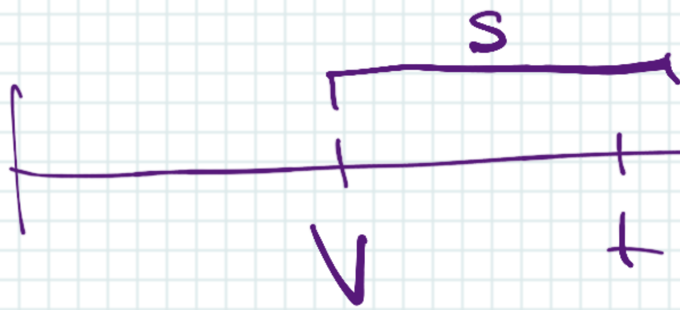
$$N(t) \xrightarrow{1-p(t)} D(t) = \#\text{ departures by } t$$

$$= \sum_{i=}^{N(t)} I\{S_i \le t - t_i\}$$



typical arrival time

$V \sim \text{unif}(0, t)$



customer who arrives at time V will still be in service at time t iff $S > t - V$.

$$p(t) = \mathbb{P}(S > t - V) = \mathbb{P}(S > V)$$

$$= \int_0^t \mathbb{P}(S > u)\,\frac{du}{t} = \frac{1}{t}\int_0^t \bar{G}(u)\,du$$

New mean of $L(t)$

is $\underbrace{\lambda t}_{\alpha} p(t) = \boxed{\lambda \int_0^t \bar{G}(u)\,du} = \alpha(t)$

$$L(t) = \sum_{i=1}^{n} I\{S_i > t - V_{(i)}\}$$

$$\Big|_{N(t)=n} \overset{\text{dist}}{=} \sum_{i=1}^{n} I\{S_i > t - V_i\} = \sum_{i=1}^{n} I\{S_i > \underline{V_i}\}$$

Binomial$(n, p(t))$

Example $\lambda = 500$

time in days

$$\boxed{G(x) = \frac{x}{4}}, \quad x \in (0, 4)$$

Unif(0,4) for service times

$E(L(2)) = ?$

$E(L(5)) = ?$

$E(L(5 \text{ years})) = ?$

$E(S) = 2$

$\mu = \frac{1}{2}$

$$\boxed{\rho = \frac{\lambda}{\mu} = 1000}$$

$$E(L(t)) = \boxed{\alpha(t)} = \lambda \int_0^t \bar{G}(u)\,du = 500 \int_0^t \bar{G}(u)\,du$$

$$\bar{G}(u) = \begin{cases} \frac{4-u}{4}, & 0 \le u < 4 \\ 0, & u \ge 4 \end{cases}$$

$$\| \quad \mathbb{P}(S > u)$$

$$\Rightarrow \int_0^t \bar{G}(u)\,du = \underbrace{\int_0^4 \bar{G}(u)\,du + \int_4^t 0}_{= \int_0^\infty} = E(S), \quad t > 4$$

$$=$$

$$d(2) = 500\int_0^2 \left(1-\frac{x}{4}\right)dx = 500\left(\frac{3}{2}\right)$$
$$= 750$$

$$d(5) = d(4) = \rho = 1000$$

$$d(t) = \rho = 1000 \quad , \quad t \geq 4$$

instead Suppose

$$G(x) = 1 - e^{-.5x}, \quad x \geq 0$$ (exponential dist., mean $= 2$)

$$\exp(.5)$$

$\mu = \frac{1}{2}$ as before

$$E(S) = 2$$

$$\alpha(t) = \lambda \int_0^t e^{-.5u}\, du = 500 \int_0^t e^{-.5u}\, du$$

$$= \frac{500}{.5}\left(1 - e^{-.5t}\right)$$

$$= 1000\left(1 - e^{-.5t}\right)$$

If $S \sim \text{unif}(1, 3)$

$$E(S) = 2$$

$$\mathbb{P}(S > u) = \begin{cases} 1, & 0 \le u < 1 \\ \frac{3-u}{2} & 1 \le u < 3 \\ 0, & u \ge 3 \end{cases}$$

$$d(t) = \lambda \int_0^t 1 \, du \quad \text{if} \quad t < 1$$

$$= \lambda t$$

Thus :

If $t = 2$, then

$$\alpha(t) = \lambda(1) + \lambda \int_1^2 \left(\frac{3-u}{2}\right) du = \lambda + \lambda \int_1^2 \left(\frac{3-u}{2}\right) du.$$

If $t = 4$, then since $t \geq 3$,

$$\alpha(t) = \alpha(3) = \lambda E(S) = \rho = 1000$$

$$(\alpha(t) = \alpha(3) = \rho, \quad t \geq 3)$$

$$\left(\text{for } t > 3, \quad \int_3^t \mathbb{P}(S > u)\, du = \int_3^t 0 \cdot du = 0\right)$$