Continuous-Time Markov Chains:
Birth & Death processes
A continuous-time stochastic process \( \{X(t) \mid t \geq 0\} \) with discrete state space \( S \subseteq \mathbb{Z} \) is called a CTMC if

\[
P(X(s+t) = j \mid X(s) = i, \exists X(u): 0 \leq u < s) \downarrow \text{future} \uparrow \text{present} \uparrow \text{past}
\]

\[
= \mathbb{P}(X(s+t) = j \mid X(s) = i) = P_{ij}(t)
\]

For each \( t \), \( (P_{ij}(t+1))_{i,j \in S} \) is a transition matrix.
Let $H_i$ denote a holding time in state $i$.

It is immediate that $H_i$ must have an exponential distribution at a rate denoted $q_i$ that only depends on $i$.

Via memoryless property: only the exp. distr. has this property.
Let $T_n =$ time (continuous) that $(X^{(t)})$ moves out of a state (a transition)

$X_n = X(T_n^+)$ state visited right after the "transition"

$X(0) = 2, \quad X(T_1^+) = 4, \quad X(T_2^+) = -1, \quad \ldots$

$X_0, \quad X_1, \quad X_2, \quad \ldots$
\((X_n)\) forms a discrete-time Markov chain with same state space \(S\) and it has a transition matrix

\[ P_{ij} = \Pr(X_{n+1} = j \mid X_n = i) \]

\((X_n)\) is called the embedded Markov Chain for \((X_{n+1})\)
So a continuous time Markov chain (CTMC), \( \mathcal{M}(X(t)) \), is a Markov chain where after a transition into state \( j \) (\( j \in J \)), \( \mathcal{M}(X(t)) \) remains in state \( j \) for an exponential amount of time \( t_j \), independent of the past before making its next transition.
\((X(t))\) is defined completely by 

\[\mathbb{P} = \{P_{ij}\}\] and a set of rates \([\lambda_i, \mu_i]\)
Example: Rat in open maze

for example

choose

\[ a_1 = 1 \]
\[ a_2 = 2 \]
\[ a_3 = 3 \]
\[ a_4 = 4 \]

\[ E(H) = \frac{1}{a_i} \]

rat spends more time in Room 1

compared to the other rooms
Question: if $X(0) = 1$

$T \overset{def}{=} \text{amount of continuous time until rat escapes.}$

$E(T) = ?$
A very simple example of a \( \text{CTMC} \) is \( \left( N(t^+) : t \geq 0 \right) \).

For a \( \mathbb{PP} \) \( \lambda \):

\[ X_n = N(t_{n+}) = N(t_n) = n, \]

\[ p_{i,i+1} = 1, \quad i \geq 0, \]
\[ N(s+t) = N(s) + (N(s+t) - N(s)) \]

future \hspace{1cm} \text{present} \hspace{1cm} \text{independent increments}

Given \( N(s) \), the future is independent of the past.
M|M|1 Queue

Service times \( (S_n) \) iid \( \exp(\mu) \)

FIFO

\[ X(t) \text{ def } \text{ the number of customers in the system at time } t \]

\[ X(t) = 4 \]

\( \mathbb{E}(X(t)) \) is a CTMC
At any time $t$,

Let $T$ denote the remaining time until next arrival and if at time $t$, $X(t) \geq 1$,

Let $S_{r}$ denote the remaining service time of the customer in service.

$T \sim \exp(\lambda)$ independent of the past and of $S_{r} \sim \exp(\mu)$. 

If $X(+) = 0$

- $P_{01} = 1$
- $H_0 \sim \exp(\lambda)$

If $X(+) = 1$

- $S_n$
- $P_{1,0} = P(S_n < T) = \frac{\lambda^n}{\lambda + n}$
- $P_{1,1} = P(T < S_n) = \frac{\lambda}{\lambda + n}$

Same for any $n \geq 1$, $X(+) = n$

- $P_{n,n-1} = \frac{\lambda^n}{\lambda + n}$
- $P_{n,n+1} = \frac{\lambda}{\lambda + n}$
\[ W = \sum_{i=1}^{N} \text{i.i.d. exp}(\lambda) \]

\[ X(t) = \# \text{ in system at } t \]

\[ X(t) = 0 \]

\[ P_{01} = 1 \]

\[ H_0 \sim \text{exp}(\lambda), \quad \theta_0 = \lambda \]

\[ H_1 \sim \text{exp}(\lambda + \mu) \]

\[ \rho_{1,0} = \frac{\theta}{\lambda + \mu} \]

\[ \rho_{1,2} = \frac{\lambda}{\lambda + \mu} \]
\[ X(+) = 2 \]

\[ S_{r_1}, S_{r_2}, S_{r_3} \text{ are i.i.d.} \]

\[ \exp(\mu) \]

\[ T \]

\[ P_{2,1} = P(\min(S_{r_1}, S_{r_2}) < T) \]

\[ \exp(2\mu) \]

\[ H_n \sim \exp(\lambda + 2\mu), \quad a_n = \lambda + 2\mu \]

\[ P_{2,3} = P(T < \min(S_{r_1}, S_{r_2})) = \frac{\lambda}{\lambda + 2\mu} = P_{n, n+1} \quad n \geq 2 \]
All these queueing examples are "Birth & Death" processes, C (TMC) for which
\[ Q = \{0, 1, 2, \ldots\} \]
and
\[ P_{i,i+1} + P_{i,i-1} = 1 \]

Birth rates \( \lambda_i, i \geq 0 \)
Death rates \( \mu_i, i \geq 0 \)
\( \mu_0 = 0 \)

\[ P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i} \]
\[ H_i = \min \{ B_{ij}, D_{ij} \} \]
\[ \exp(\lambda_i) \quad \exp(\mu_i) \]
\[ X(+) = i \]
\[ M/m \] \quad S \overset{\text{def}}{=} \frac{\lambda}{\mu}

\[ p_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{x(s)=j} ds \]

long-run proportion of time the chain spends in state \( j \)

Want \( \{ p_j : j \geq 0 \} \) to form a prob. dist.,

"limiting prob. dist." of \((X(t))_{t \geq 0}\) C.T.M.C.
Rate equations

For each $j \geq 0$

Rate out of state $j$ = Rate into state $j$

For $M|M|1$

$\lambda P_0 = \nu P_1$

$(\lambda + \nu)P_j = \lambda P_{j-1} + \nu P_{j+1}$ for $j \geq 1$

$(\lambda + \nu)P_1 = \lambda P_0 + \nu P_2$

$\lambda P_1 + \lambda P_0 = \lambda P_0 + \nu P_2 \Rightarrow \lambda P_1 = \nu P_2$
\[ P_j = \frac{1}{n} P_{j+1}, \quad j \geq 0 \]

rate from \( j \) to \( j+1 \) = rate from \( j+1 \) to \( j \)

also must have \( \sum_{j=0}^{\infty} P_j = 1 \)

\[ P_0 = \frac{1}{n} P_0 = s P_0 \]
\[ P_1 = s P_0 = s^2 P_0 \]
\[ P_2 = s P_1 = s^3 P_0 \]
\[ \vdots \]
\[ P_j = s^j P_0, \quad j \geq 0 \]
\[ p_j = 1 \quad \iff \quad p_0 \sum_{j=0}^{\infty} s^j = 1 \]

\[ \iff s < 1 \] in which case

\[ p_0 \cdot \frac{1}{1-s} = 1 \quad \Rightarrow \quad p_0 = 1 - s \]

\[ p_j = s^j (1-s), \quad j \geq 0 \]
\[
\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} X(s) \, ds = \sum_{j=0}^{\infty} j p_j = \frac{\lambda}{1-\rho}
\]

\( P_0 = \text{long-run prop. of time the system is empty} = 1-\rho \)

\( 1 - P_0 = S = \text{long-run proportion of time the system is busy} \)
Queue
Special case of $\text{M}/\text{M}/\infty$ queue
$G(r) = 1 - e^{-ux}$

We already know that the limiting dist. is Poisson($\lambda$)

$p_j = \frac{e^{-\lambda} \lambda^j}{j!}, \quad j \geq 0$
\[
\begin{align*}
\text{Rate} & = \lambda \\
\mathbb{P}(j+1) & \rightarrow \mathbb{P}(j) \\
\lambda \mathbb{P}_0 & = \nu \mathbb{P}_1 \\
\lambda \mathbb{P}_1 & = 2 \nu \mathbb{P}_2 \\
\lambda \mathbb{P}_2 & = 3 \nu \mathbb{P}_3 \\
\vdots & \vdots \\
\lambda \mathbb{P}_j & = (j+1) \nu \mathbb{P}_{j+1}
\end{align*}
\]

\[
\sum_{j=0}^{\infty} \mathbb{P}_j = 1 \
\Rightarrow \quad P_0 \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} = 1 \\
\Rightarrow P_0 e^{-\lambda} = 1 \\
\Rightarrow P_0 = e^\lambda
\]
\[ \Rightarrow P_j = e^{-5} \frac{5^j}{j!} \quad j \geq 0 \]

\( \text{Poisson}(5) \)