IEOR 4106 lec 14

Continuous-Time Markov Chains;

Birth & Death processes

a continuous-time stochastic process $\{X(t) ; t \geq 0\}$ with <u>discrete</u> state space $\mathcal{S} \subseteq \mathbb{Z}$ is called a CTMC if

$$\mathbb{P}\big(\underbrace{X(s+t)=j}_{\text{future}} \mid \underbrace{X(s)=i}_{\underline{\text{present}}}, \underbrace{\{X(u) : 0 \leq u < s\}}_{\text{past}}\big)$$

$$= \mathbb{P}(X(s+t)=j \mid X(s)=i) = P_{ij}(t)$$

For each t, $(P_{ij}(t))_{i,j \in \mathcal{S}}$ is a transition matrix

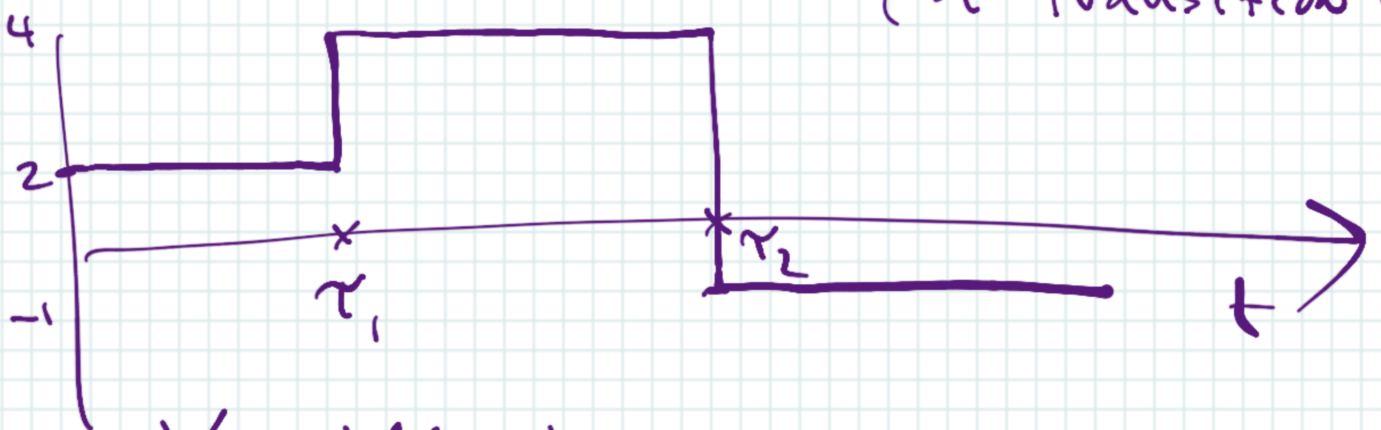
Let H_i denote a holding time in state $i \in \mathcal{S}$.

It is immediate that

H_i must have an exponential distribution at a rate denoted q_i that only depends on i

Via memoryless property: only the exp. distr. has this property

Let τ_n = time (continuous) that $(X(t))$ moves out of a state (a transition)



$X_n = X(\tau_n+)$ = state visited right after the "transition"

$X(0) = 2, \quad X(\tau_1+) = 4, \quad X(\tau_2+) = -1 \quad \ldots$

$\quad\; \| \qquad\qquad\quad \| \qquad\qquad\quad \|$

$\quad X_0 \qquad\qquad\; X_1 \qquad\qquad\; X_2$

(X_n) forms a discrete-time Markov chain with same state space $\mathcal{S}$

and it has a transition matrix

$$P_{ij} = \mathbb{P}(X_{n+1}=j \mid X_n=i)$$

(X_n) is called the embedded MC for $\{X(t)\}$

So a <u>Continuous Time</u> MC

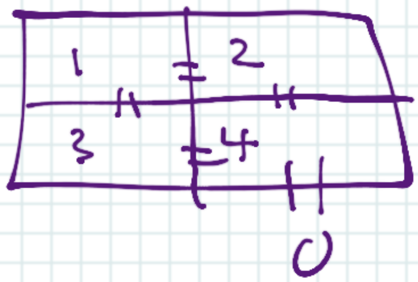
CTMC $(X(t))$

is a Markov chain which after a transition into state $j \in \mathcal{S}$, remains in state j for an exponential amount of time H_j independent of the past before making its next transition.

$\{X(t)\}$ is defined completely by

$P = (P_{ij})$ and a set of rates $\{a_i : i \in \mathcal{S}\}$

Example: Rat in open maze



for example
choose $a_1 = 1$
$a_2 = 2$
$a_3 = 3$
$a_4 = 4$

$$E(H_i) = \frac{1}{a_i}$$

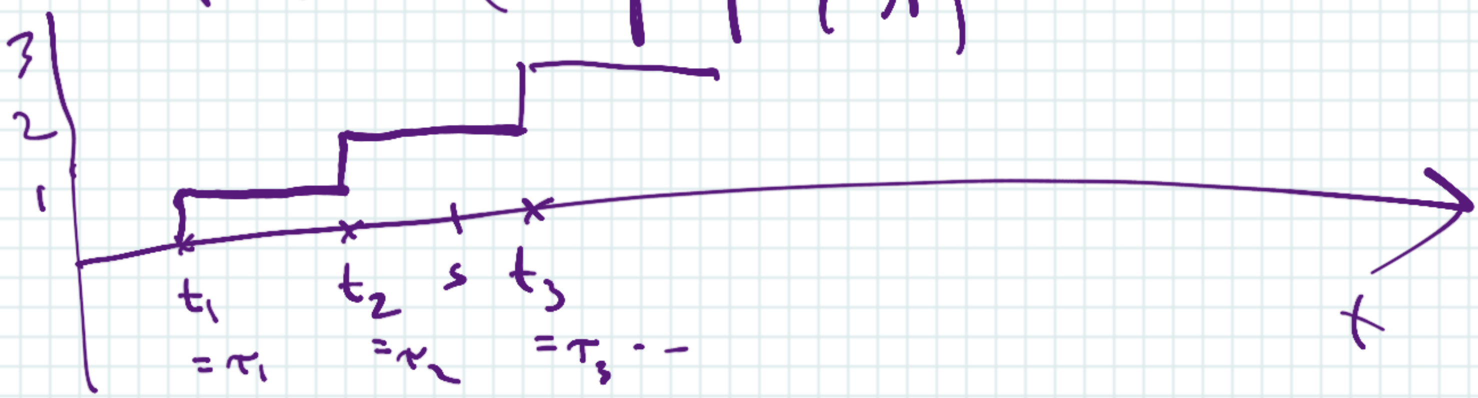
rat spendes more time in Room 1
compared to the other rooms

Question: if $X(0) = 1$,

$T \stackrel{\text{def}}{=}$ amount of continuous time until rat escapes.

$$E(T) = ?$$

A very simple example of a CTMC is $(N(t) : t \geq 0)$ for a $PP(\lambda)$



$$X_n = N(\tau_n +) = N(t_n^+) = n$$

$$P_{i,i+1} = 1 \quad , \quad i \geq 0$$

$$N(s+t) = N(s) + \underbrace{\big(N(s+t) - N(s)\big)}$$

future present independent increments

Given $N(s)$, the future is independent of the past

$M|M|1$ queue

Service times (S_n) iid $\exp(\mu)$

FIFO

$PP(\lambda)$

$X(t) \overset{\text{def}}{=}$ the number of customers in the system at time t

$X(t) = 4$

$\{X(t)\}$ is a CTMC

At any time t,

Let T denote the remaining time until next arrival

and if at time t $X(t) \geq 1$,

Let S_r denote the remaining service time of the customer in service

$T \sim \exp(\lambda)$ independent of the past and of $S_r \sim \exp(\mu)$

$a_0 = \lambda$

$a_i = \lambda + \nu \quad i \geq 1$

$H_i \overset{dist}{=} \min(S_\nu, T)$

if $X(t) = 0$

$P_{01} = 1 \quad H_0 \sim \exp(\lambda)$

$\sim \boxed{T}$

if $X(t) = 1$

S_ν

$\boxed{P_{1,0} = \mathbb{P}(S_\nu < T) = \frac{\nu}{\lambda + \nu}}$

T

$P_{1,2} = \mathbb{P}(T < S_\nu) = \frac{\lambda}{\lambda + \nu}$

Same for any $n \geq 1$, $X(t) = n$

S_n

T

$\boxed{P_{n,n-1} = \frac{\nu}{\lambda+\nu}, \quad P_{n,n+1} = \frac{\lambda}{\lambda+\nu}}$

M|M|2 queue

(S_n) iid $\exp(\mu)$

$X(t) = \#$ in system at t

$\uparrow$ $pp(\lambda)$

$X(t)=0$

$P_{01}=1$

$H_0 \sim \exp(\lambda)$, $q_0 = \lambda$

$X(t)=1$

Sr

T

$P_{1,0} = \dfrac{\mu}{\lambda+\mu}$

$P_{1,2} = \dfrac{\lambda}{\lambda+\mu}$

$H_1 \sim \exp(\lambda+\mu)$

$X(t) = 2$

Sr_1  Sr_2

Sr_1, Sr_2 are iid $\exp(\mu)$

T

$H_n \sim \exp(\lambda + 2\mu)$, $n \geq 2$, $a_n = \lambda + 2\mu$

$$P_{2,1} = \mathbb{P}(\underbrace{\min(Sr_1, Sr_2)}_{\exp(2\mu)} < T) = \frac{2\mu}{\lambda + 2\mu} = P_{n,n-1}, \quad n \geq 2$$

$$P_{2,3} = \mathbb{P}(T < \min(Sr_1, Sr_2)) = \frac{\lambda}{\lambda + 2\mu} = P_{n,n+1}, \quad n \geq 2$$

All these queueing examples are "Birth & Death" processes, CTMCs for which

$\mathcal{S} \subseteq \{0, 1, 2, \cdots\}$

and $P_{i,i+1} + P_{i,i-1} = 1$

Birth rates λ_i, $i \geq 0$

Death rates μ_i, $i \geq 0$ $\quad \mu_0 = 0$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$X(t) = i$

$$H_i = \min\{B_i, D_i\}$$

$\exp(\lambda_i)$ $\exp(\mu_i)$

M/M/1 $\quad \rho \overset{def}{=} \frac{\lambda}{\mu}$

$$P_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s)=j\}\, ds$$

= long-run proportion of time the chain spends in state j

want $\{P_j : j \geq 0\}$ to form a prob. dist,
"limiting prob. dist" of $(X(t))$ CTMC

rate equations

for each $j \geq 0$

rate out of state j = rate into state j

for M|M|1

$j=0$

$\rightarrow \lambda P_0 = \mu P_1$

$j=1$

$(\lambda+\mu)P_1 = \lambda P_0 + \mu P_2$

$\lambda P_1 + \cancel{\lambda P_0} = \cancel{\lambda P_0} + \mu P_2 \rightarrow \lambda P_1 = \mu P_2$

$(\lambda+\mu)P_j = \lambda P_{j-1} + \mu P_{j+1}$

$j \geq 1$

$$\lambda P_j = \mu P_{j+1}, \quad j \geq 0 \qquad \boxed{\rho = \frac{\lambda}{\mu}}$$

rate from $j \mapsto j+1$ = rate from $j+1 \mapsto j$

also must have $\displaystyle\sum_{j=0}^{\infty} P_j = 1$

$P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$

$P_2 = \rho P_1 = \rho^2 P_0 \cdots \quad \boxed{P_j = \rho^j P_0, \quad j \geq 0}$

$$\sum_{j=0}^{\infty} P_j = 1 \iff$$

$$P_0 \sum_{j=0}^{\infty} \rho^j = 1$$

$\iff \boxed{\rho < 1}$ in which case

$$P_0 \frac{1}{1-\rho} = 1 \rightarrow P_0 = 1 - \rho$$

geometric dist. !

$$\boxed{P_j = \rho^j (1-\rho), \; j \geq 0}$$

$$\rho < 1$$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)\,ds = \sum_{j=0}^{\infty} j\,P_j = \frac{\rho}{1-\rho}$$

$P_0 =$ long-run prop. of time the system is empty $= 1-\rho$

$1 - P_0 = \rho =$ long-run proportion of time the system is busy

$\boxed{M|M|\infty}$ queue

special case of $M|G|\infty$ queue

$G(x) = 1 - e^{-\nu x}$

We already know that the limiting dist. is Poisson(ρ)

$$P_j = e^{-\rho} \frac{\rho^j}{j!}, \quad j \geq 0$$

$\rho = \frac{\lambda}{\mu}$

$$\underset{j \mapsto j+1}{\text{rate}} = \underset{j+1 \mapsto j}{\text{rate}}$$

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$\lambda P_2 = 3\mu P_3$$
$$\vdots$$
$$\lambda P_j = (j+1)\mu P_{j+1}$$

$P_1 = \rho P_0$

$P_2 = \frac{\rho}{2}\rho P_0 = \frac{\rho^2}{2} P_0$

$\vdots$

$$\boxed{P_j = \frac{\rho^j}{j!} P_0}, \quad j \geq 1$$

$$\sum_{j=0}^{\infty} P_j = 1 \iff P_0 \sum_{j=0}^{\infty} \frac{\rho^j}{j!} = 1 \Rightarrow P_0 e^{\rho} = 1$$

$$\Rightarrow \boxed{P_0 = e^{-\rho}}$$

$$\Longrightarrow \quad p_j = e^{-\rho} \frac{\rho^j}{j!}, \quad j \geq 0$$

Poisson(ρ)