IEOR 4106 lec 15

CTMCs continued:

1) more on Birth & Death processes

2) PASTA

CTMCs

$$\mathbb{P}(X(s+t)=j \mid X(s)=i, \{X(u): 0 \le u < s\})$$
$$= \mathbb{P}(X(s+t)=j \mid X(s)=i) = P_{ij}(t)$$

Markov property

$\Rightarrow$ holding times in each state j

$\sim \exp(q_j)$

$j \in \mathcal{S} \subseteq \mathbb{Z}$

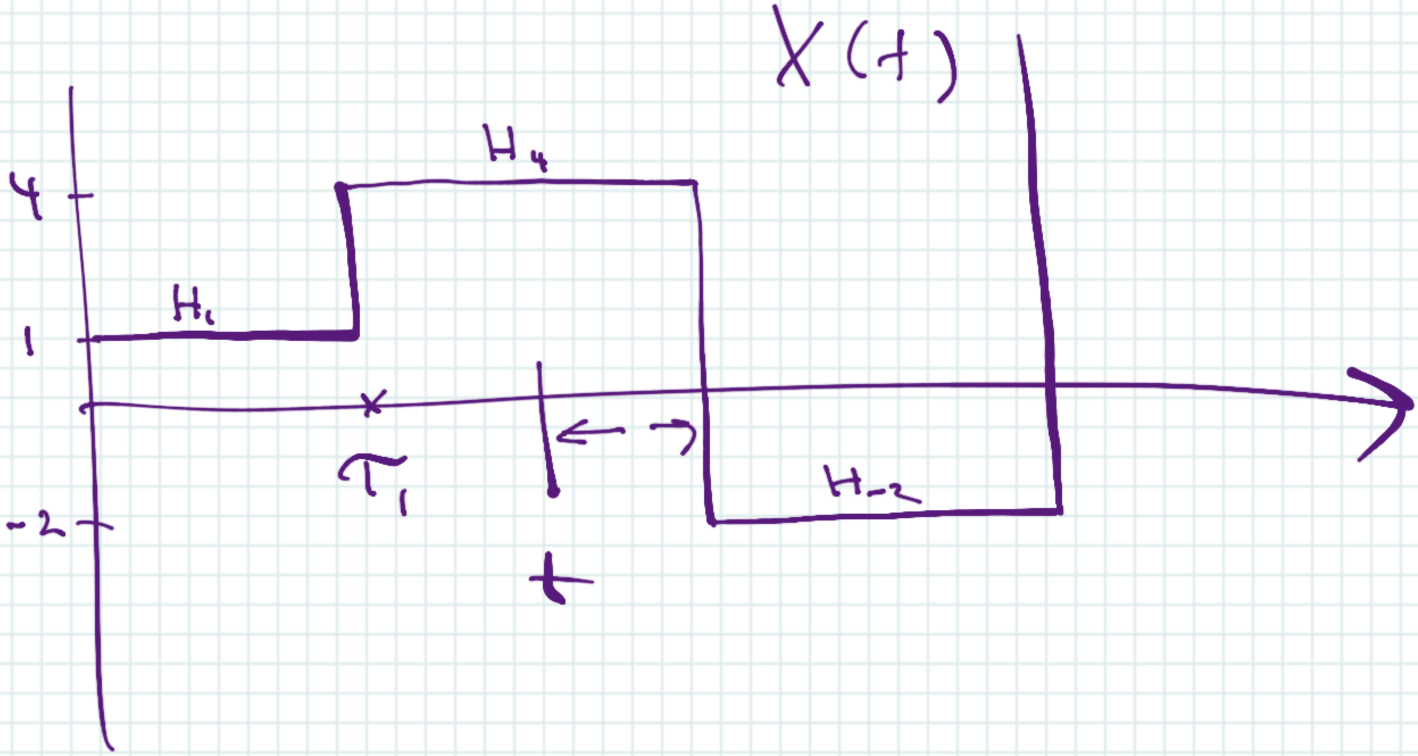
X_n = n^{th} state visited by $\{X(t)\}$

forms a discrete-time MC on $\mathcal{S}$

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

"embedded MC"

$$\boxed{\underline{P = (P_{ij})}_{i,j \in \mathcal{S}} \qquad \underline{\{q_j : j \in \mathcal{S}\}} \quad X(0) = i}$$



$X(0) = 1$

rates for determining

limiting distributions

$$P_j \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s)=j\}\,ds \quad \text{wp1}$$

(we want $P_j > 0,\ j \in \mathcal{S}$ and $\sum_{j\in\mathcal{S}} P_j = 1$)

"limiting dist."

$(X(t))$ is irreducible if (X_n) is so.

Similarly with communication classes:

$$\mathcal{S} = \bigcup_k C_k$$

where the C_k are disjoint communication classes (same as those of (X_n))

rate out of state j = rate into state j $\quad j \in \mathcal{S}$

for an irreducible CTMC we obtain "Global Balance Equations" using the Markov Property

$$q_j P_j = \sum_{i \neq j} q_i P_i P_{ij} \qquad \text{for all } j \in \mathcal{S}$$

Birth & Death processes are the special case of CTMC when $\mathcal{S} = \{0, 1, 2, \dots\}$

or subset of

and $P_{i,i+1} + P_{i,i-1} = 1 \quad i \in \mathcal{S}$

Birth : up by 1

Death : down by 1 $\quad (\mu_0 = 0)$

$H_i \sim \exp(a_i) \quad a_i = \lambda_i + \mu_i$

$H_i = \min(B_i, D_i) \quad B_i \sim \exp(\lambda_i) \quad D_i \sim \exp(\mu_i)$

independent

Example: M|M|1 queue

pp(λ)

Service times iid

$(S_n) \sim \exp(\mu)$

$$\rho \overset{\text{def}}{=} \lambda/\mu$$

$$\underline{P_{01} = 1}$$

$\lambda_i = \lambda, \quad i \geq 0$

$\mu_i = \mu, \quad i \geq 1$

$(\mu_0 = 0)$

$$P_{i,i+1} = \frac{\lambda}{\lambda+\mu} \qquad i \geq 1$$

$$P_{i,i-1} = \frac{\mu}{\lambda+\mu}$$

For a Birth & Death process the Global Balance Equations reduce to

"rate from $j \mapsto j+1$ = rate from $j+1 \mapsto j$"

Yielding "Birth & Death Balance Equations"

$$\boxed{\lambda_j P_j = \mu_{j+1} P_{j+1}, \quad j \geq 0}$$

M/M/1

$\lambda P_j = \mu P_{j+1}$ if $\boxed{\rho = \lambda/\mu < 1}$

$P_j = \rho^j (1-\rho)$, $j \geq 0$

Geometric Dist.

$P_{01} = 1$

$P_{i,i+1} = \dfrac{\lambda}{\lambda+\mu}$, $P_{i,i-1} = \dfrac{\mu}{\lambda+\mu}$

(X_n) $\quad$ " p $\quad$ " $1-p$

M | M | ∞ queue

PP(λ) $\xrightarrow{}$ exp(ν)

(special case of M|G|∞ queue)

$$\lambda P_j = (j+1)\nu P_{j+1}$$

$j \geq 0$

$\nu_j = j\nu \quad j \geq 0$

$\lambda_j = \lambda$

$$P_0 = e^{-\rho}$$

$$P_j = e^{-\rho} \frac{\rho^j}{j!}, \quad j \geq 0$$

Poisson(ρ) dist.

PASTA: Poisson Arrivals See Time Averages

For $X(t)$ = number of customers in a queueing model at time t

$\{t_n\}$ (customers arrival times)

$$P_j \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s)=j\}\,ds$$

= long-run prop. of time there are j customers in the system

$$\pi_j^a \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} I\{X(t_n^-)=j\}$$

= long-run prop. of <u>arrivals</u> who find j in system

PASTA: if $\{t_n\}$ is a Poisson process

then (under mild conditions)

$$P_j = \pi_j^a, \quad j \geq 0$$

M|M|1 loss Queue

B & D process

$$\mathcal{S} = \{0, 1\}$$

$\boxed{0}$ $\quad$ iid $(S_n) \sim \exp(\mu)$

No line is allowed: any arrival finding the server busy is **lost**.

$\uparrow$ PP(λ)

$$X(t) = \begin{cases} 1 & \text{if server is busy at } t \\ 0 & \text{if server is idle at } t \end{cases}$$

P_1 = long-run prop. time the server is busy

P_0 = long-run prop. time the server is idle.

$0 < \rho < \infty$ $\quad \lambda_0 = \lambda$ $\quad (\lambda_1 = 0)$

$\mu_1 = \mu$ $\quad \lambda_j = 0,\ j \geq 1)$

$(\mu_0 = 0)$

$\lambda P_0 = \mu P_1$

$\boxed{P_0 + P_1 = 1}$

$P_1 = \rho P_0$

$P_0(1+\rho) = 1,$ $\quad \boxed{P_0 = \frac{1}{1+\rho} \quad P_1 = \frac{\rho}{1+\rho}}$

$$P_0 = \pi_0^a$$

$$\boxed{P_1 = \pi_1^a = \frac{\rho}{1+\rho}}$$ via PASTA

long-run prop. of arrivals (from $PP(\lambda)$) who are lost $= \pi_1^a = P_1 = \boxed{\frac{\rho}{1+\rho}}$

M/M/c loss model

iid $\sim \exp(\mu)$ service times

No line

$S = \{0, 1, \cdots, c\}$

$PP(\lambda)$

$\lambda_j = \lambda, \quad 0 \le j \le c-1$

$\mu_j = \underline{j\mu}, \quad 0 \le j \le c$

$c = 2$

$\mathcal{S} = \{0, 1, 2\}$

PP(λ)

$\lambda P_0 = \mu P_1$ $\qquad P_1 = \rho P_0$

$\lambda P_1 = 2\mu P_2$ $\qquad P_2 = \frac{\rho^2}{2} P_0$

$\underline{P_0 + P_1 + P_2 = 1}$ $\qquad P_0\left(1 + \rho + \frac{\rho^2}{2}\right) = 1$

$$P_0 = \left(1 + \rho + \frac{\rho^2}{2}\right)^{-1}$$

$$P_1 = \rho P_0, \quad P_2 = \frac{\rho^2}{2} P_0$$

$$C = 3$$

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$\lambda P_2 = 3\mu P_3$$

$$P_1 = \rho P_0$$
$$P_2 = \frac{\rho^2}{2} P_0$$
$$\boxed{P_3 = \frac{\rho^3}{3!} P_0}$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

$$P_0 \left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{3!}\right) = 1$$

$$P_0 = \left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{3!}\right)^{-1}$$

for any $c \geq 1$,

$\Pi_c^a = P_c$ via PASTA

$=$ Prop. of arrivals who are lost

Machine breakdown model

$$\boxed{\begin{aligned}\lambda P_0 &= \mu P_1\\ \lambda P_1 &= 2\mu P_2\\ \lambda P_2 &= 3\mu P_3\end{aligned}}$$

$\exp(\lambda)$
Service
(Repair)
times

3 "up"
machines
Lifetimes
iid $\exp(\mu)$

$$\mathcal{S} = \{0, 1, 2, 3\}$$

$$\boxed{X(t) = \text{number of } \underline{\text{up}} \text{ machines at time } t}$$

B & D process

$\lambda_0 = \lambda \qquad \mu_0 = 0$

$\lambda_1 = \lambda \qquad \mu_1 = \mu$

$\lambda_2 = \lambda \qquad \mu_2 = 2\mu, \quad \mu_3 = 3\mu$

$(\lambda_j = 0, \; j \geq 3)$

Same equations as for $M|M|3$ loss model

$$P_0 = \left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{3!}\right)^{-1}$$

$$P_j = \frac{\rho^j}{j!} P_0 \qquad 0 \le j \le 3$$

General B & D Equations

$$\lambda_j P_j = \mu_{j+1} P_{j+1}, \quad j \geq 0$$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} P_0$$

$$\vdots$$

$$P_j = P_0 \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}$$

$$\sum P_j = 1$$

$$P_0 \left(\sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i} + 1 \right) = 1$$

$$P_0 = \frac{1}{\sum_{j=1}^{\infty}\left(\prod_{i=1}^{j}\frac{\lambda_{i-1}}{\mu_i}\right) + 1}$$