IEOR 4106 lec 16

CTMCs, PASTA

"$\ell = \lambda w$" (Little's Law)

Global Balance Equations for CTMCs

$\{P_j\}$ limiting Probabilities

rate out of state j   rate into state j

$$a_j P_j = \sum_{i \neq j} a_i P_i P_{ij} \qquad H_i \sim exp(a_i) \quad i \in \mathcal{S}$$

$$\boxed{P_j \overset{def}{=} \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s)=j\} ds} \qquad j \in \mathcal{S}$$

$$T_{jj} = \begin{cases} \text{return time to state } j \\ \mid X(0) = j \\ \overset{\text{def}}{=} \infty \quad \text{if no return} \end{cases}$$

$$\left(T_{jj} \begin{cases} = \min\{n \geq 1 : X_n = j \mid X_0 = j\} \\ \overset{\text{def}}{=} \infty \quad \text{if no return} \end{cases} \right)$$

$$\geq H_j$$

(we are interested in the case when $\mathbb{P}(T_{jj} < \infty) = 1$

<u>recurrence of</u> state j)

$$1 = \mathbb{P}(T_{jj} < \infty) \Longleftrightarrow \mathbb{P}(T_{jj} < \infty) = 1$$

j is recurrent

transient otherwise

Π

Pos. rec. in Cont. time

($X(t)$)

$$\boxed{E(T_{jj}) < \infty}$$

If j is recurrent, then by the Markov property visits to state j form a renewal process $\boxed{\{t_n(j)\}}$

$t_n(j)$ = time at which $(X(t))$ visits state j for n^{th} time $n \geq 1$

(i.i.d) interarrival times $\sim T_{jj}$

$(N_j^+(t))$ counting process

$\{Y_m : m \geq 1\}$

$Y_m = t_m(j) - t_{m-1}(j)$

$N_j^-(t)$ = number of times $(X(t))$ departs state j during $(0,t]$

$N_j^+(t)$ = number of times $(X(t))$ enters state j during $(0,t]$

they differ by at most 1 for each t

$$\lim_{t\to\infty} \frac{N^+(t)}{t} = \boxed{\frac{1}{E(T_{jj})}} \quad \text{wp1}$$

ERT

$$= \lim_{t\to\infty} \frac{N^-(t)}{t} \quad \text{wp1}$$

"rate $(X(t))$ enters j = rate $(X(t))$ leaves state j"

$$= \frac{1}{E(T_{jj})}$$

$$P_j = \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s)=j\}\,ds$$

$$\frac{1}{t}\sum_{m=1}^{N_j^-(t)} H_j(m) \le \frac{1}{t}\int_0^t I\{X(s)=j\}\,ds \le \frac{1}{t}\sum_{m=1}^{N_j^+(t)} H_j(m)$$

$$\left(\frac{N_j^+(t)}{t}\right)\left(\frac{1}{N_j^+(t)}\sum_{m=1}^{N_j^+(t)} H_j(m)\right)$$

$\boxed{H_j(m) \overset{dist}{=} H_j}$

iid holding times per visit to j

$t\to\infty$ ERT $\dfrac{1}{E(T_{jj})}$

SLLN $t\to\infty$ $E(H_j)$

$\exp(q_j)$

$E(H_j) = \dfrac{1}{q_j}$

$$= \frac{E(H_j)}{E(T_{jj})} = \frac{\frac{1}{q_j}}{E(T_{jj})}$$

left-hand side is the same

$$\Rightarrow P_j = \frac{\frac{1}{q_j}}{E(T_{jj})} \qquad j \in \mathcal{S}$$

$$\Rightarrow q_j P_j = \frac{1}{E(T_{jj})} = \text{rate out/in to state } j$$

$P_{01} = 1$

$$\boxed{a_j P_j = \sum_{i \neq j} a_i P_i P_{ij}} \qquad j \in \mathcal{S}$$

B & D process $\quad a_j = \lambda_j + \mu_j \quad j \geq 0$

$$P_{i,i+1} + P_{i,i-1} = 1$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$\lambda_0 P_0 = (\lambda_1 + \mu_1) P_1 \cdot P_{10}$$

$$= \lambda_1 + \mu_1 P_1 \frac{\mu_1}{\lambda_1 + \mu_1} = \mu_1 P_1$$

$$\boxed{\lambda_0 P_0 = \mu_1 P_1}$$

$P_{2,1}$

$$(\lambda_1+\mu_1)P_1 = \lambda_0 P_0 + (\mu_2+\lambda_2)P_2 \quad \frac{\mu_2}{\lambda_2+\mu_2}$$

$$\boxed{(\lambda_1+\mu_1)P_1 = \lambda_0 P_0 + \mu_2 P_2}$$

$$\Rightarrow \quad \lambda_1 P_1 = \mu_2 P_2$$

$$(\lambda_2+\mu_2)P_2 = \lambda_1 P_1 + \mu_3 P_3$$

$$\Rightarrow \quad \lambda_2 P_2 = \mu_3 P_3 \quad \cdots$$

$$\boxed{\lambda_j P_j = \mu_{j+1} P_{j+1} \quad j \geq 0}$$

Birth & Death Balance Equations

"rate from $j \to j+1$ = rate from $j+1 \to j$"

$$\boxed{\lambda_j P_j = \mu_{j+1} P_{j+1}, \quad j \geq 0}$$

For a queueing model

$X(t)$ = number of customers in the system at time t

arrivals form a point process $\{t_n\}$

$$p_j = \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s)=j\}\,ds$$

$$\pi_j^a = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} I\{X(t_n^-)=j\}$$

= long-run proportion of customers who arrive finding j already in system

PASTA (Poisson Arrivals See Time Averages)

if $\{t_n\}$ is a $PP(\lambda)$ (Poisson)

then $p_j = \pi_j^a \quad , \; j \in \mathcal{S}$

If arrivals are <u>not</u> Poisson, result does <u>not</u> hold in general

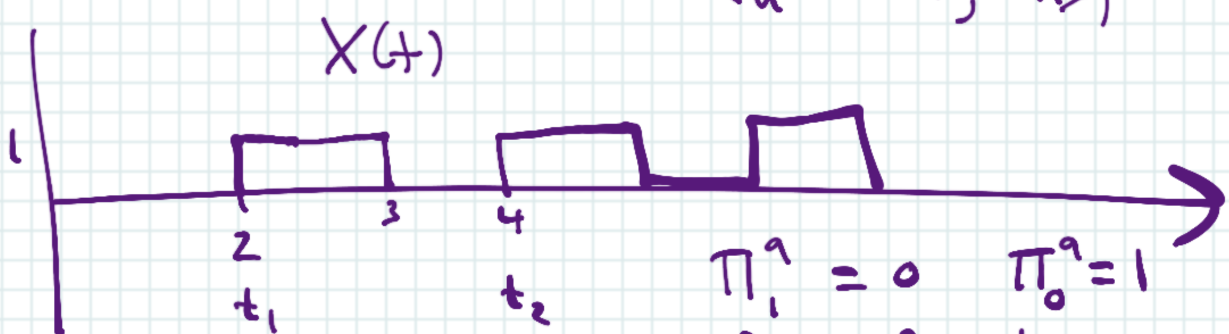
Counterexample: $(X(t))$

☐ Single-Server Queue

$S_n \equiv 1, \quad n \geq 1$

interarrival times $\equiv 2$

$t_n = 2n, \quad n \geq 1$



$\pi_1^a = 0 \qquad \pi_0^a = 1$

$P_1 = P_0 = \frac{1}{2}$

Proof of PASTA for M/M/1 Queue

rate at which
$X(t)$ moves $j \mapsto j+1$

$$\cancel{\lambda}\pi_j^a = \cancel{\lambda}P_j$$

Birth and Death Balance Equations

basic principles

$$\boxed{\pi_j^a = P_j \qquad j \geq 0}$$

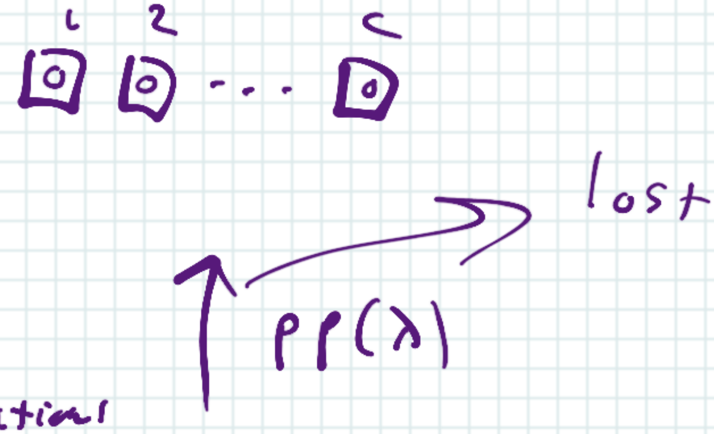
Nice application:

Erlang's Loss formula

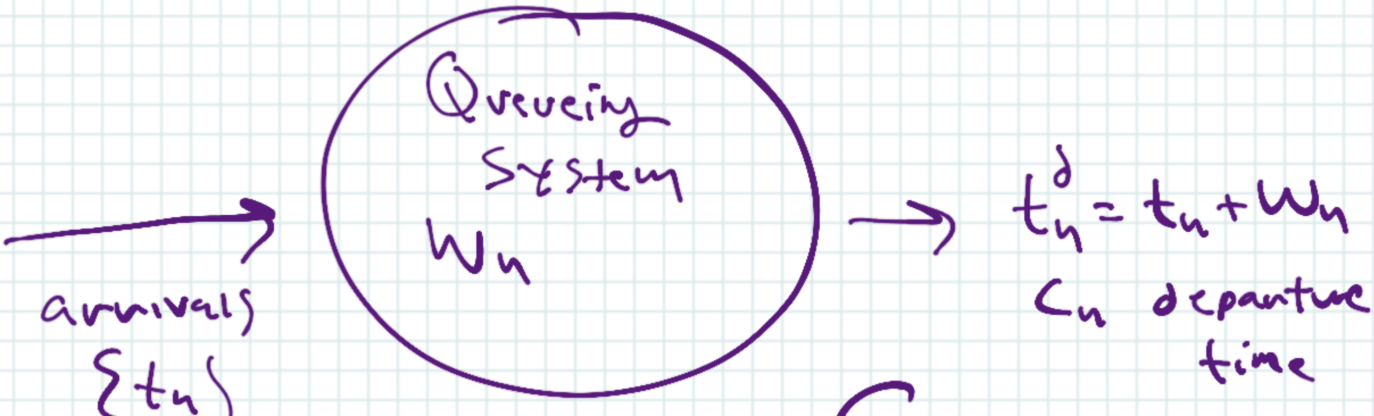
M/M/c loss queue

$$\pi_c^a \overset{\text{PASTA}}{=} p_c$$

Solved using Balance equations



"$\ell = \lambda w$" Little's Law



nth customer C_n

arrives at time n
and enters the system

has a sojourn time $= W_n \geq 0$

$=$ total time spent in the system

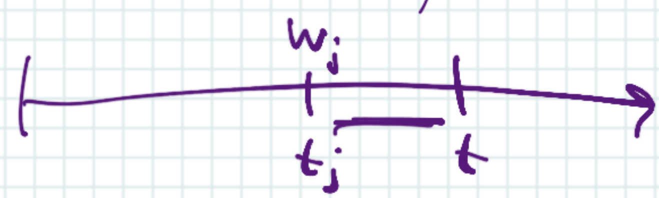
Defined if they exist

$$\lambda \stackrel{\text{def}}{=} \lim_{t \to \infty} \frac{N(t)}{t}$$ arrival rate

$$w \stackrel{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} w_j$$ average sojourn time

$L(t)$ = number of customers in the system at time t

$$= \sum_{j=1}^{N(t)} I\{w_j > t - t_j\}$$



$$\ell \stackrel{\text{def}}{=} \lim_{t \to \infty} \frac{1}{t} \int_0^t L(s)\, ds$$

time average
number of customers
in the system

"$\ell = \lambda w$": If λ and w exist
and are $< \infty$, then
ℓ exists and

$$\ell = \lambda w.$$

Example: SEAS

undergraduates

$\lambda = 1500$ year

$w = 4$ years

$\ell = \lambda w = 6000$ students

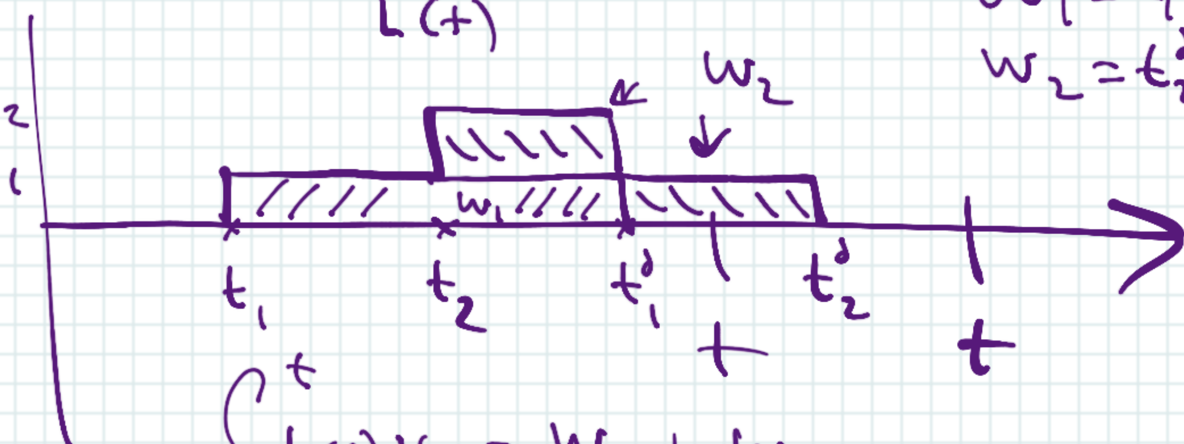
Proof: $\lim_{t \to \infty} \frac{1}{t} \int_0^t L(s)\,ds$

area under $L(s)$ $[0, t]$ $= \int_0^t L(s)\,ds \approx \sum_{j=1}^{N(t)} W_j$

$L(t)$

$W_1 = t_1^d - t_1$

$W_2 = t_2^d - t_2$



$\int_0^t L(s)\,ds = W_1 + W_2$

$$\frac{1}{t}\sum_{\{j \,:\, t_j^d \le t\}} W_j \le \frac{1}{t}\int_0^t L(s)\,ds \le \frac{1}{t}\sum_{j=1}^{N(t)} W_j$$

$$\left(\frac{N(t)}{t}\right)\left(\frac{1}{N(t)}\right)\sum_{j=1}^{N(t)} W_j$$

$$\downarrow \qquad \downarrow$$

$$\lambda \qquad w$$

λ, w exist and $< \infty$ by assumption

Left-hand side also converses to λw (more complicated proof)

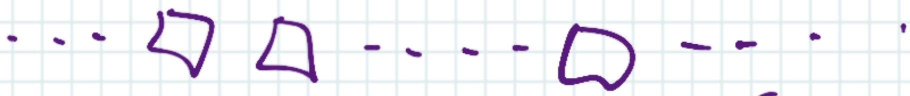
So the result is complete.

$$\boxed{l = \lambda w}$$

applications

∞ Server Queue

(M|G|∞ queue)



Service times $\{S_n\}$

$\{t_n\}$ λ

$W_n = S_n$

$$\frac{1}{\mu} \overset{def}{=} w = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} S_j$$

average service time

$$\rho \overset{def}{=} \lambda/\mu$$

if λ, $\frac{1}{\nu}$ exist and $< \infty$,

then ℓ exists

$$\ell = \rho$$

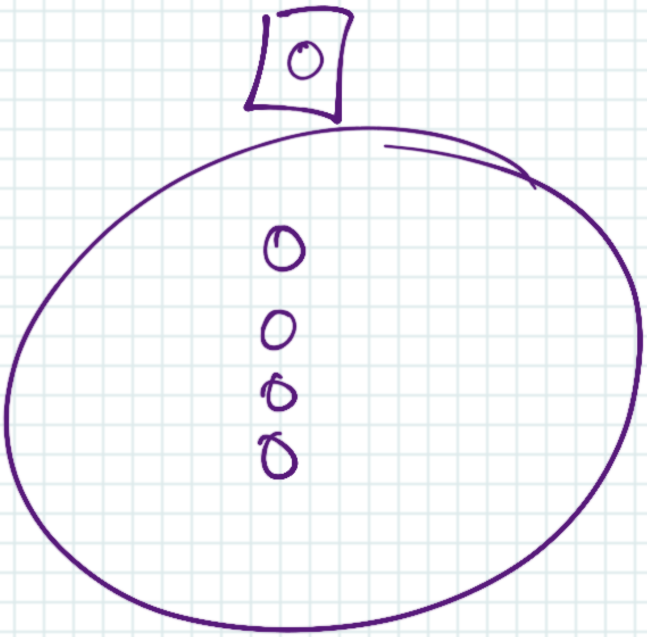
Recall M|G|∞ queue $\ell = \rho$

$X(t) \sim \text{Poisson}(\alpha(t))$

$\lim_{t \to \infty} \alpha(t) = \rho$ $\quad E(X(t)) \to \rho$

$X(t) \Longrightarrow \text{Poisson}(\rho)$

Same answer!



"$Q = \lambda d$"

average number in line

average delay in line