IEOR 4106 lec 17

1) More applications of "$\ell = \lambda w$" (Little's Law)

2) forwards/Backwards rec. time for renewal processes.
the inspection paradox

$$\ell = \lambda w$$

$$\ell \stackrel{def}{=} \lim_{t \to \infty} \frac{1}{t} \int_0^t L(s)\,ds$$ = average number of customers in the system

$$w \stackrel{def}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} W_j$$ = average <u>Sojourn</u> time.

$$\lambda \stackrel{def}{=} \lim_{t \to \infty} \frac{N(t)}{t}$$ = arrival rate of customers entering the system

If both λ and w exist and are $< \infty$, then ℓ exists and $\ell = \lambda w$

examples:

arrival rate λ

1) ∞-server queue (t_n) $(N(t))$

$W_j = S_j$, service times

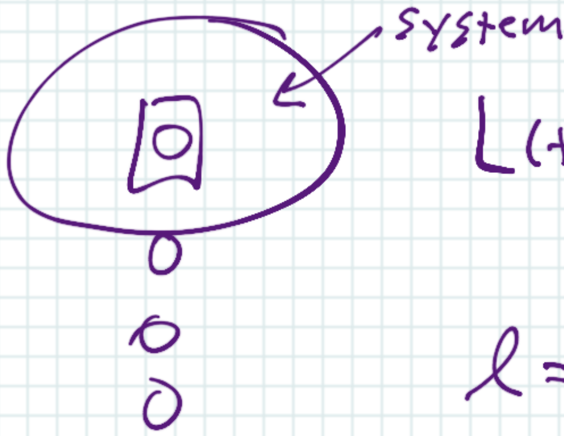
$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} W_j$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} S_j = \frac{1}{\mu}$$

$$\rho = \lambda/\mu$$

$$l = \lambda w = \lambda \cdot \frac{1}{\mu} = \rho$$

2) Single-Server Queue (Let's use the Server as the system)



$$L(t) = \begin{cases} 1 & \text{if server is busy} \\ 0 & \text{if server is idle} \end{cases}$$

ℓ = long-run proportion of time the server is busy

$\uparrow \{t_n\}$ λ = arrival rate

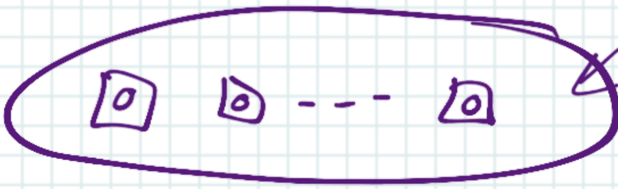
$\left(\int_0^t L(s)\,ds = \right.$ amount of time during $(0, t]$ that the server is busy$\left.\right)$

$W_j = S_j$, $w = \frac{1}{\mu}$

$\ell = \lambda \cdot \frac{1}{\mu} = \rho$ $\quad(\rho < 1)$ $\quad(\lambda < \mu)$

$1 - \rho =$ long-run prop time the server is idle

3) c-server queue (all c-servers are the system)



$L(t) =$ number of busy servers at t $\in \{0, 1, \dots, c\}$

$w_j = S_j$

$w = \frac{1}{\mu}$

$\{t_n\}$ rate λ

$\lambda < c\mu$

$(\rho < c)$

$\rho = \lambda \cdot \frac{1}{\mu} = \rho$

$= $ long-run average number of busy servers

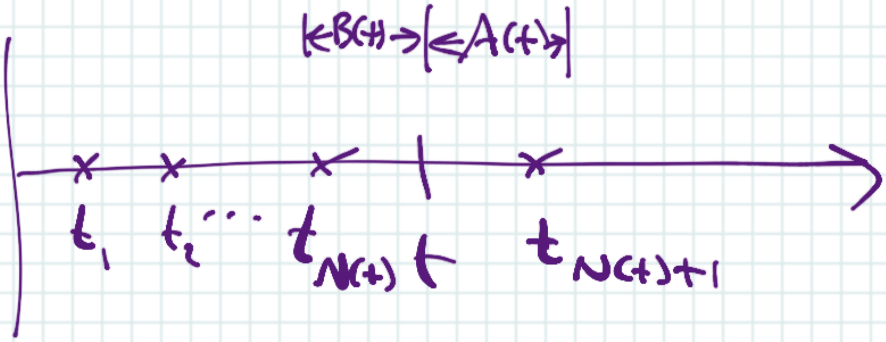
$0 < \rho < c$

Renewal process $\{t_n\}$

iid interarrival times

$$X_n = t_n - t_{n-1}$$

$$0 < E(X) < \infty$$



$$\lambda \stackrel{\text{def}}{=} \frac{1}{E(X)}$$

arrival rate

$$= \lim_{t \to \infty} \frac{N(t)}{t} \quad \text{wp1}$$

ERT

$$N(t) = \max\{n : t_n \le t\}$$

$$A(t) = t_{N(t)+1} - t, \quad B(t) = t - t_{N(t)}$$

$A(t)$ is called the forward (excess) recurrence time at time t

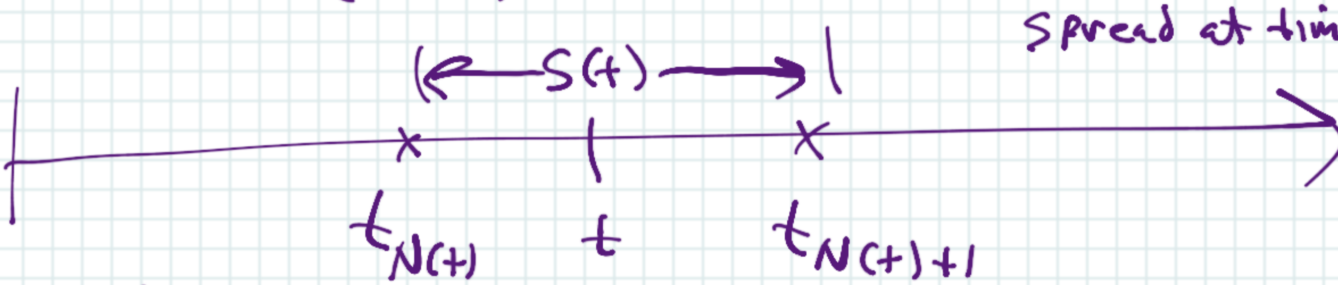
$B(t)$ backwards recurrence time at time t

(asc)

$= X_{N(t)+1}$

$S(t) = B(t) + A(t)$

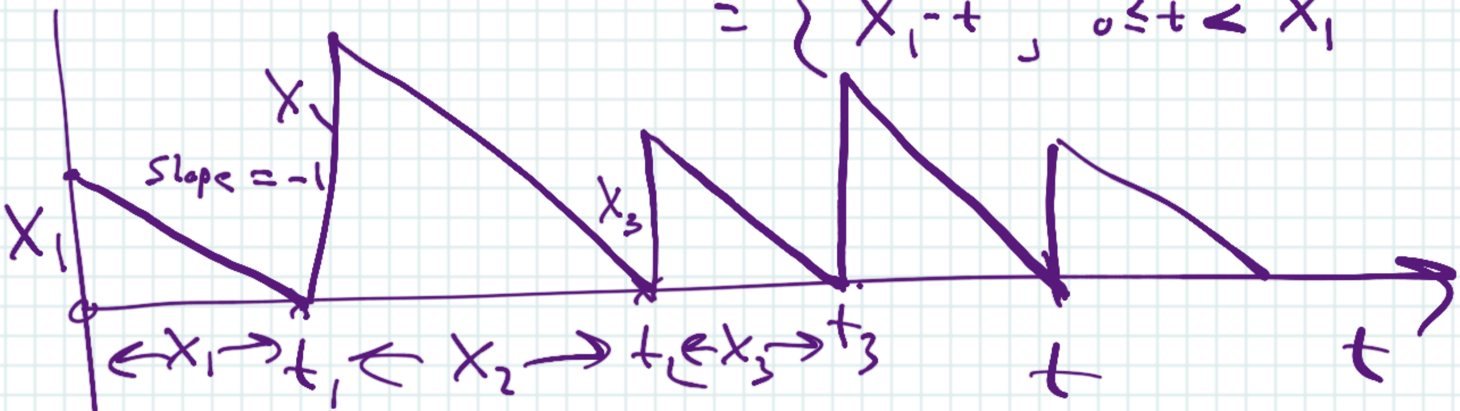
spread at time t



$X_{n+1} = t_{n+1} - t_n$

$$A(t) = t_{N(t)+1} - t$$

$$= \{ X_1 - t \,, \quad 0 \le t < X_1$$



$$\lim_{t \to \infty} \frac{1}{t} \int_0^t A(s)\,ds$$

$$\int_0^t A(s)\,ds \approx \frac{X_1^2}{2} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2}$$

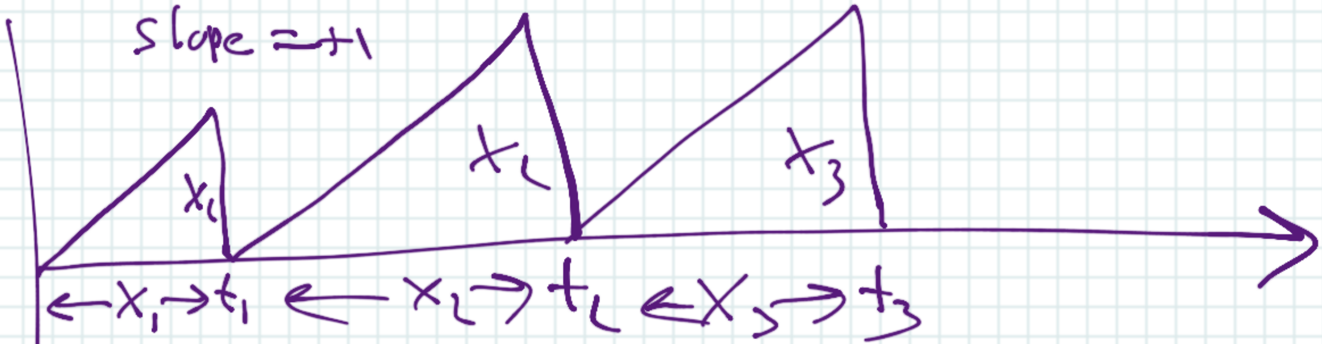
$$\frac{1}{t}\sum_{i=1}^{N(t)}\frac{X_i^2}{2} \le \frac{1}{t}\int_0^t A(s)\,ds \le \frac{1}{t}\sum_{i=1}^{N(t)+1}\frac{X_i^2}{2}$$

$$= \left(\frac{N(t)}{t}\right)\left(\frac{1}{N(t)}\sum_{i=1}^{N(t)}\frac{X_i^2}{2}\right)$$

$$\frac{N(t)}{t} \xrightarrow{\text{ERT}} \lambda = \frac{1}{E(X)} \qquad \frac{1}{N(t)}\sum_{i=1}^{N(t)}\frac{X_i^2}{2} \xrightarrow{\text{SLLN}} \frac{E(X^2)}{2}$$

$$\boxed{\frac{E(X^2)}{2E(X)}}$$

$$B(t) = t - t_{N(t)}$$



$$\lim_{t \to \infty} \frac{1}{t} \int_0^t B(s)\,ds = \frac{E(X^2)}{2E(X)} \quad \text{wp1}$$

also

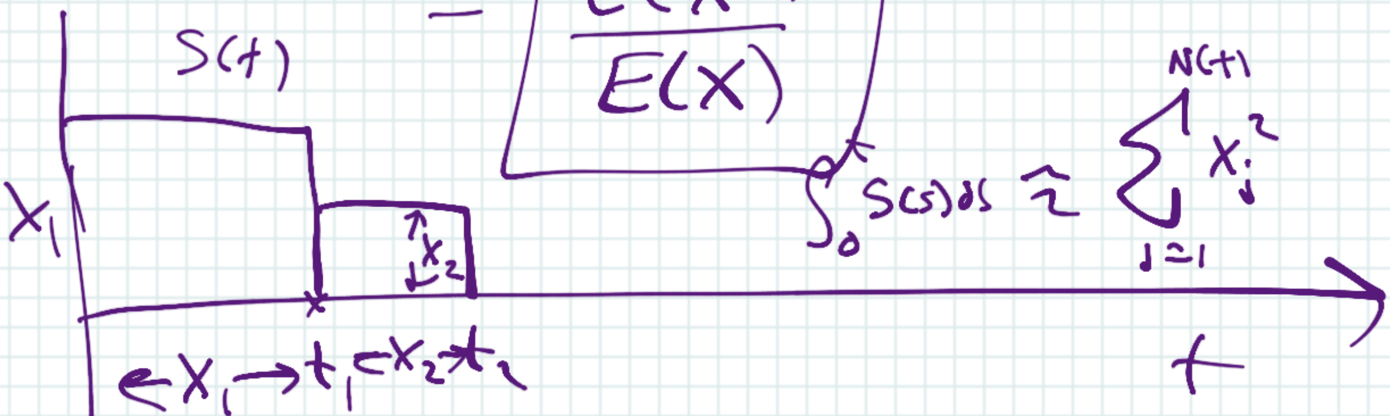
$$S(t) = t_{N(t)+1} - t_{N(t)}$$

$$= B(t) + A(t)$$

spread at time t

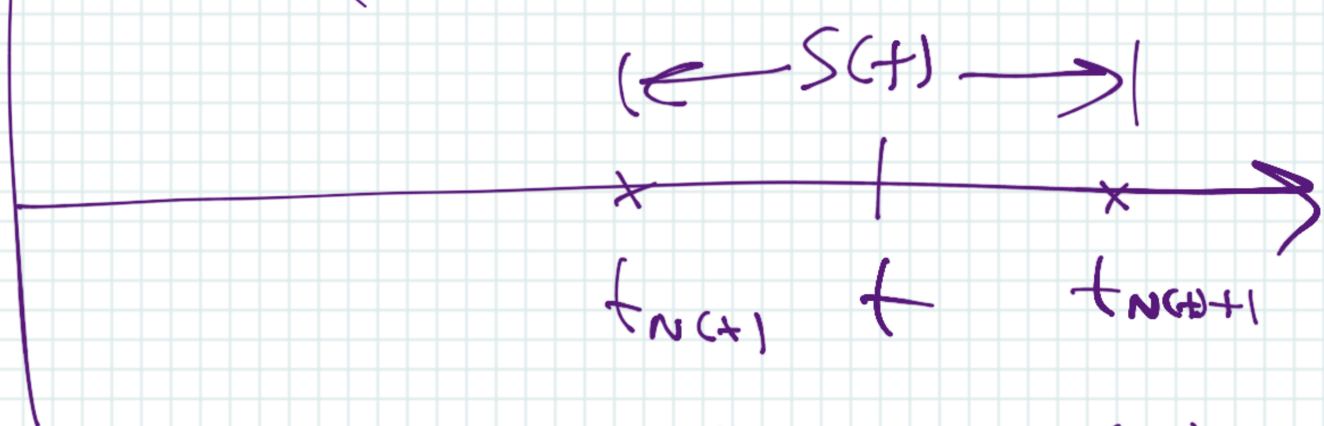
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds = \boxed{\frac{E(X^2)}{E(X)}}$$

$$\int_0^t S(s)\,ds \approx \sum_{i=1}^{N(t)} X_i^2$$



$$S(t) = X_1 \qquad 0 \le t < X_1$$

assume $PP(\lambda)$ $\qquad$ $S(t) = B(t) + A(t)$



$A(t) \sim \exp(\lambda)$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S(s)\,ds = \frac{E(X^2)}{E(X)} = \frac{2}{\lambda} = 2E(X)$$

$$\lambda = \frac{1}{E(X)}, \quad E(X^2) = \frac{2}{\lambda^2} \quad (X \sim \exp(\lambda))$$

$$Var(X) = E(X^2) - E^2(X) > 0$$

$$\Rightarrow \quad E(X^2) \geq E^2(X)$$

$$\Rightarrow \quad \boxed{\frac{E(X^2)}{E(X)}} \geq \frac{E^2(X)}{E(X)} = E(X)$$

Inspection Paradox

Average interval you land in is larger than the average of (X_n), $E(X)$

$$\frac{E(X^2)}{E(X)} \geq E(X)$$

$\{X_n\} = \{1, 2, 1, 2, 1, 2, \ldots$

$$\frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow[n\to\infty]{} \frac{1}{2}(1) + \frac{1}{2}(2) = \frac{3}{2}$$

in time t, $\frac{2}{3}$ of the time a length 2 is in progress

$\frac{1}{3}$ of the time a length 1 is in progress

$$\frac{2}{3}(2) + \frac{1}{3}(1) = \frac{5}{3} > \frac{3}{2}$$

average length of the interarrival time you land in $= 5/3$

$$\mathbb{P}(X=0) = .99$$
$$\mathbb{P}(X=10) = .01$$

$$E(X) = .99(0) + (.01)(10) = 0.1$$

$$S(t) = 10$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds = 10 = \frac{E(X^2)}{E(X)} = \frac{1}{.1} = 10$$

$$E(X^2) = .99(0^2) + .01(10^2) = (.01)(100) = 1$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I\{A(s)>x\}\,ds, \quad x\ge 0$$

$A(s) = X_1 - s, \quad 0\le s < X_1$

$$\int_0^t I\{A(s)>x\}\,ds \overset{?}{=} \sum_{j=1}^{N(t)} (X_j - x)^+$$

$$\int_0^{t_1 = X_1} I\{X_1 - s > x\}\,ds = \begin{cases} X_1 - x & \text{if } X_1 > x \\ 0 & \text{if } X_1 \le x \end{cases}$$

$a^+ = \max(0, a)$
pos. part of a

$$\frac{1}{t}\int_0^t I\{A(s)>x\}\,ds$$

$$\approx \frac{1}{t}\sum_{j=1}^{N(t)}(X_j-x)^+$$

$$= \left(\frac{N(t)}{t}\right)\underbrace{\left(\frac{1}{N(t)}\right)\sum_{j=1}^{N(t)}(X_j-x)^+}$$

$t\to\infty$ $\xrightarrow[\text{ERT}]{} \lambda=\frac{1}{E(X)}$ $\xrightarrow[\text{SLLN}]{} E(X-x)^+$

$$\bar{F}_e(x) \overset{\text{def}}{=} \frac{E(X-x)^+}{E(X)}$$

$F(x) = \mathbb{P}(X \le x)$

$x \ge 0$

$\bar{F}(x) = 1 - F(x)$

$= \mathbb{P}(X > x)$

Let $Y = (X-x)^+ \ge 0$

$$E(Y) = \int_0^\infty \mathbb{P}(Y > y)\,dy = \int_0^\infty \mathbb{P}((X-x)^+ > y)\,dy$$

$$= \int_0^\infty \mathbb{P}(X-x > y)\,dy$$

$$= \int_0^\infty \mathbb{P}(X > y+x)\,dy = \boxed{\int_x^\infty \mathbb{P}(X > y)\,dy}$$

$$\overline{F_e}(x) = \frac{\int_x^\infty \bar{F}(y)\,dy}{\int_0^\infty \bar{F}(y)\,dy} \xrightarrow[x \to \infty]{} 0$$

$$F_e(x) = \frac{\int_0^x \bar{F}(y)\,dy}{\int_0^\infty \bar{F}(y)\,dy} \xrightarrow[x \to \infty]{} 1 = \frac{E(X)}{E(X)}$$

Defines a prob distn called the <u>equilibrium</u> distn.

$$\lambda = \frac{1}{E(X)} \qquad \left(0 < E(X) < \infty \right)$$

$$F_e(x) = \lambda \int_0^x \bar{F}(y)\,dy, \quad x \geq 0$$

always continuous

$$\left(\int_0^\infty n x^{n-1} \bar{F}(x)\,dx = E(X^n) \right)$$

$$\boxed{f_e(x) = F_e'(x) = \lambda \bar{F}(x)}$$

Let X_e denote a rv dist. as F_e

$$E(X_e) = \int_0^\infty x f_e(x)\,dx = \frac{\lambda}{2} \int_0^\infty 2x \bar{F}(x)\,dx \qquad n=2$$

$$= \frac{\lambda}{2} E(X^2) = \frac{E(X^2)}{2E(X)}$$

example: $X \sim \exp(x)$

$$\bar{F}(x) = e^{-\lambda x}$$

$$f_e(x) = \lambda e^{-\lambda x}$$

$$\sim \exp(\lambda)$$

example $\mathbb{P}(X = 1) = 1 \qquad \lambda = 1$

$$\bar{F}(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x \geq 1 \end{cases}$$

$$f_e(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

unif $(0,1)$ density