IEOR 4106 lec 18

Renewal Reward Thm

more on the inspection Paradox
Last lecture

\[ A(+) = t_{N(+) + 1} - t \]
\[ B(+) = t - t_{N(+)} \]
\[ S(+) = A(+) + B(+) \]

Renewal processes

forwards (excess) rec. time

backwards (age) rec. time

spread

\[ F(x) = P(X \leq x) = t_{N(x) + 1} - t_{N(x)} \]

Stochastic processes in continuous time, \( t \geq 0 \)

\[ \lim_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t} A(s) = \frac{\mathbb{E}(X^2)}{2 \mathbb{E}(X)} = \frac{\lambda \mathbb{E}(X^2)}{2} \]
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t B(s) \, ds = \frac{\text{E}(X^2)}{2\text{E}(X)} \quad \text{wp1 also,}
\]

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(s) \, ds = \frac{\text{E}(X^2)}{\text{E}(X)} \geq \text{E}(X)
\]

"The average size of the interarrival time you land in over time is larger than the E(X)."

\(X\) denotes a copy of the original iid \(X_n\) interarrival times.
\[ F_e(x) = \lambda \int_0^x f(y) \, dy \quad \text{equilibrium dist. of } F, \quad x \geq 0 \]

\[ = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}\{A(s) \leq x\} \, ds \]

\[ F'_e(x) = \lambda F(x) = f_e(x) \quad \text{density function of the equilibrium dist.} \]
**Renewal Reward Theorem**

Taxi drops off passengers at times $\{t_n; n \geq 1\}$ that form a renewal process with interarrival times

$$X_n = t_n - t_{n-1}, \quad n \geq 1$$

$$0 < E(X) < \infty$$

$$\lambda = \frac{1}{E(X)}$$

$R_n$ = $\text{\# earned at time } t_n$

($\text{you pay driver at the end of your ride}$)

$$R(+) = \sum_{j=1}^{n(t_\tau)} R_j = \text{Total } \# \text{ earned by time } \tau$$
We assume that the vectors \( \{ (X_j, R_j) : j \geq 1 \} \) are i.i.d.

\[
\frac{R(t)}{t} \quad \xrightarrow{t \to \infty} \quad \text{long-run rate that the taxi driver earns} \quad \frac{E(R)}{E(X)}
\]

\[
\frac{1}{t} \sum_{j=1}^{N(t)} R_j = \frac{(N(t))}{t} \left( \frac{1}{N(t)} \right) \sum_{j=1}^{N(t)} R_j \xrightarrow{\text{EAT}} \lambda = \frac{1}{E(t)} \quad \sqrt{E(R)} \xrightarrow{\text{CLT}} \frac{E(R)}{E(X)}
\]
Expected value over a cycle divided by an expected cycle length

Renewal Reward Thm: Suppose $R_j$ is earned during $X_j$ (any which way)

0 < $E(X) < \infty$, \[ E(\mid R \mid) < \infty \]

$R(t) =$ total $\$ earned by time $t$

\[ \lim_{t \to \infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)} \ \text{wp1} \]
Proof: If $R_j \geq 0$, then

$$R_+ = \max \{0, R_j\}$$

and

$$R_- = -\min \{0, R_j\}$$

If $E R_j^+ < \infty$ and $E R_j^- < \infty$ (general case), then

$$E(R) = \frac{E(R_+)}{E(R_+)}$$

with

$$R_+ = \sum_{j=1}^{N_G} R_j$$

and

$$R_- = \sum_{j=1}^{N_G} R_j$$
Recall that ERT also has that $E(N(t))$ approaches $\lambda$ as $t \to \infty$ and $N(t)$ approaches $\lambda$ as $t \to \infty$. \[\left(\frac{N(t)}{t} \to \lambda\right)\] with probability 1.
It also holds that

\[ \lim_{t \to \infty} \frac{E(R_{t+1})}{E(X)} = \frac{E(R)}{E(X)} \]

(Proof is beyond the scope of this course.)
Applications \[ A(s) = X_1 - s, \ 0 \leq s < X_1 \]

\[ v(x) = A(x) \] is the rate at time \( t \) at which \( \$ \) is earned.

\[ \int_0^x A(s) \, ds = \int_0^t r(s) \, ds \quad R = R_t \]

\[ R_t = \int_0^{x_t} r(s) \, ds = \int_0^{x_1} r(s) \, ds \]

\[ E(R) = \frac{E(X^4)}{250} \]

\[ E(X) = \frac{E(X^2)}{2} \]
\[ v^+(s) = B(s) \]

\[ v(s) = S(s) \]

\[ R_1 = \int_0^{x_1} ds = \frac{x_1^2}{2} \]

Same \( \lim_{t \to \infty} \frac{R(t)}{t} = \frac{EX^2}{2E(x)} \)

\[ R_1 = \int_0^{x_1} ds = x_1 \int_0^{x_1} ds = x_1^2 \]

\[ \frac{E(R)}{E(x)} = \frac{E(x^2)}{E(x)} \]

\[ B(s) = \delta_j, \quad 0 \leq s < x_1 \]
Car Replacement problem with "T" policy

New cars cost $C_1$, and have iid lifetimes \( \{V_j : j \geq 1 \} \)

with a cont. dist.

\[
F(x) = \mathbb{P}(V \leq x)
\]

\(f(x)\) density

Cost $C_2$ if car breaks down (dis)

For fixed time $T > 0$

We do the following to save $
Keep car until either it dies or to time $T$, which ever happens first.
If car is still alive at $T$
  give it away (free) and buy a new car. If car breaks down before $T$ pay $C_2$
and buy new car at $C_1$

$R(t) = \text{total }$A spent by $t$

$$\lim_{t \to \infty} \frac{R(t)}{t} = ?$$
\[ X_j = \min \{ V_{i,j} : i \leq j \} \]

\[ B_j = C_1 + C_2 I\{ V_j \leq T \} \]

\[ E(R) = C_1 + C_2 P(V \leq T) = C_1 + C_2 F(T) \]

\[ E(X) = E(\min(V_{j,T})) = \int_0^\infty \int_0^\infty \int_0^\infty \left[ \int_0^\infty F(y) \, dy \right] \, dy \]
Also can be computed as

\[ E(X) = \int_0^T x f(x) dx + T \mathbb{E}(f(T)) \]

\[ \frac{E(R)}{E(X)} = \frac{C_1 + C_2 F(T)}{\int_0^T F(y) dy} = g(T) \]

Try to find the value of \( T \) that minimizes this cost \( g(T) \) (set \( g'(T) = 0 \) and solve.)
\[ V \sim \text{unif}(0, 10) \]

For example: Suppose

\[ F(t) = \begin{cases} 
\frac{t}{10}, & 0 \leq t < 10 \\
1, & t \geq 10 
\end{cases} \]

\[ c_1 = 3, \quad c_2 = \frac{1}{2} \]

\[ \frac{c_1 + c_2}{\int_0^T (1 - \frac{t}{10}) \, dt} = \frac{3 + \frac{1}{2}}{T - T/20} = S(T) \]

\[ S'(T) = 0 \Leftrightarrow T^2 + 120T - 1200 = 0 \]

\[ T = 9.25 \]
Train dispatching problem

Passengers arrive to a train platform as a renewal process \( \{s_n\} \)
interarrival times of passengers

\[ T_n = s_n - s_{n-1} \quad (s_0 \equiv 0) \]

Train departs when the \( N^{th} \) passenger arrives

Train company incurs a "waiting cost" of

\[ \frac{N \cdot C}{\text{unit time}} \text{ that } N \text{ passengers are waiting} \]

\[ + \quad A \cdot K \text{ for dispatching (cost)} \]
iid cycle lengths distributed as

\[ X = T_1 + \cdots + T_N = S_N \]

\[ E(X) = N \cdot E(T) = \frac{N}{\lambda} \quad (\lambda = \frac{1}{E(X)}) \]

\[ R = 0 < T_1 < T_2 + 2 < T_3 + \cdots < (N-1) < T_N + K \]

\[ E(R) = c \cdot E(T) \left( 1 + 2 + \cdots + (N-1) \right) + K \]

\[ = c \cdot E(T) \frac{(N-1)(N)}{2} + K \]

\[ \sqrt{\frac{V}{N}} \]
\[ \frac{R(t)}{t} \rightarrow \frac{E(R)}{E(X)} \]

\[ = S(N) = \frac{c(N-1)}{2} + \frac{kn}{N} \]

minimize cost:

\[ S'(N) = 0 \]

\[ c/2 - \frac{kn}{N^2} = 0 \quad \Rightarrow \quad N = \sqrt[3]{\frac{2kn}{c}} \]

\[ S''(N) = \frac{2kn}{N^3} > 0 \quad \text{yes a minimum.} \]
If $N$ is not an integer, choose the smallest one above, and largest one below and see which is better.

\[
\begin{pmatrix}
K = 6, & N = 1, & C = 2 \\
N = \sqrt{6} \approx 2.45 \\
2 & 3 & g(2) = g(3)
\end{pmatrix}
\]
Back to Inspection Paradox:

for a renewal process:
    For each $t$
    \[ P(S(t) > x) \geq P(X > x) \text{ for all } x \geq 0 \]
    "Stochastically larger"

\[ \Rightarrow E(S(t)) \geq E(X) \text{ for all } t \]