1) More on Renewal Reward

Thm, inspection paradox

2) Random Number

Generators (See posted Lecture)

3) Generating Normal (c^2) rvs

Generating a Poisson rV
Inspection Paradox

for renewal processes:

\[ \Pr(S(t) > x) \geq \Pr(X > x), \quad x > 0 \]

"Stochastically Larger"

\[ t_{\text{NC}} \]

\[ S(t) = t_{\text{NC}+1} - t_{\text{NC}} \]

\[ X_{\text{NC}+1} \]
\[ \Rightarrow \int_0^\infty 1_{P(S(t) > x)} \, dx \geq \int_0^\infty 1_{P(X > x)} \, dx \]

\[ \Rightarrow E(S(t)) \geq E(X) \]

for all \( t \).
Renewal Reward Theorem

\[ \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} S_c(s) \, ds = \frac{E(R)}{E(X)} \quad \text{w.p. 1} \]

\[ R = R_1 = \int_{0}^{x_1} S_c(s) \, ds = \int_{0}^{x_1} \, ds = x_1 \cdot x_1 = x_1^2 \]

\[ R_j = x_j^2, \quad j \geq 1 \]
CTMC $X(t+1)$ irreducible and pos. rec.,

$0 < E(T_{jj}) < \infty \quad j \in \mathcal{S}$

$$p_j = \frac{\frac{1}{\lambda_j}}{E(T_{jj})} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{I}(X(s) = j') ds$$

Via Renewal Reward:

$$X_n = \text{times between visits to state } j$$

$$P_{nj} = \text{n}^{th} \text{ holding time in state } j$$

$$\left( \frac{E(R_j)}{E(X)} = \frac{E(H_j)}{E(T_{jj})} \right)$$
Simulation

Inverse transform method for generating a rv $X$ with a given (desired) CDF:

$$F(x) = P(X \leq x), \quad x \in \mathbb{R}$$

If the inverse function is explicitly known, $F^{-1}(y), \quad y \in (0, 1)$ then simply set

$$X = F^{-1}(U)$$
1) What if $F^{-1}$ is not known?

2) Are there better algorithms than using $F^{-1}$?

1) $N(n, \sigma^2)$ (Normal dist.)

- Mean $\mu$, Variance $\sigma^2$
- $Z \sim N(0, 1)$ (standard unit normal)

$Z$ has density $\theta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$

CDF $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$, $x \in \mathbb{R}$
\[ \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} \]

\[ X = \sigma Z + \mu \sim N(\mu, \sigma^2) \]

\[ \Rightarrow \text{Sufficient to set an algorithm for simulating } Z \sim N(0, 1) \]
Polar Method

Suppose $x, y$ are i.i.d. $N(0, 1)$

$$\begin{align*}
X &= R \cos(\Theta) \\
Y &= R \sin(\Theta)
\end{align*}$$

$$R^2 = x^2 + y^2$$
$$\Theta = \text{arc tan} \left( \frac{y}{x} \right)$$

Fact: \( R^2 \sim \exp \left( \frac{1}{2} \right) \)
\( \Theta \sim \text{unif}(0, 2\pi) \)

Independent \( I \)
\( S: R^2 \mapsto R^2 \)

(Proof based on Jacobian matrix determinant of mapping $S(x, y) = (x^2 + y^2 \text{arc tan} \left( \frac{y}{x} \right))$ etc.)
Algorithm (Polar method)

1) generate \( U_1, U_2 \)

Set \( R^2 = -2 \ln(U_1) \sim \exp(\frac{1}{2}) \)
Set \( \theta = 2\pi U_2 \sim \text{unif}(0, 2\pi) \)

Set \( X = \sqrt{R^2} \cos(\theta) \)
\( Y = \sqrt{R^2} \sin(\theta) \)

Then \( X, Y \) are iid \( N(0,1) \) r.v.s.
Poisson dist. \( N \sim \text{Poisson}(d) \)

\[
P(N = k) = e^{-d} \frac{d^k}{k!}, \quad k \geq 0
\]

Observation: \( N \sim N(1) \)

when \((N(t+1) | \text{is the})\) continuous process of \(PP(d)\)

\begin{align*}
(N(t) & \sim \text{Poisson}(dt) \\
\text{for all } t \geq 0, \quad \text{choose } t = 1
\end{align*}

Thus it suffices to give an algorithm for simulating \(N(1)\)
Algorithm:

Set $M = \min \{ n \geq 1 : u_1 u_2 \ldots u_n < e^{\frac{1}{d}} \}$

= $\min \{ n \geq 1 : |N(u_1 u_2 \ldots u_n)| < e^{\frac{1}{d}} \}$

= $\min \{ n \geq 1 : (-\frac{1}{d})N(u_1) + \ldots + (-\frac{1}{d})N(u_n) > 1 \}$

Recall: $-\frac{1}{d} \ln(u) \sim \exp(\lambda)$

$pp(d) : \{ t_n \}$

$t_n = (-\frac{1}{d} \ln(u_1) + \ldots + (-\frac{1}{d} \ln(u_n))

= \min \{ n \geq 1 : t_n > 1 \} = N(d) + 1$
\[ N = N(i) = N(i)+1 - 1 = M - 1 \]

Algorithm is thus:

Set \( M = \min \{ n \geq 1 : U_1 \cdots U_n < e^{-x} \} \)

Set \( N = M - 1 \).

Then \( N \sim \text{Poisson}(\lambda) \).