IEOR 4106 lec 19

1) more on Renewal Reward Thm, inspection paradox

2) Random Number Generators (See posted Lecture Notes)

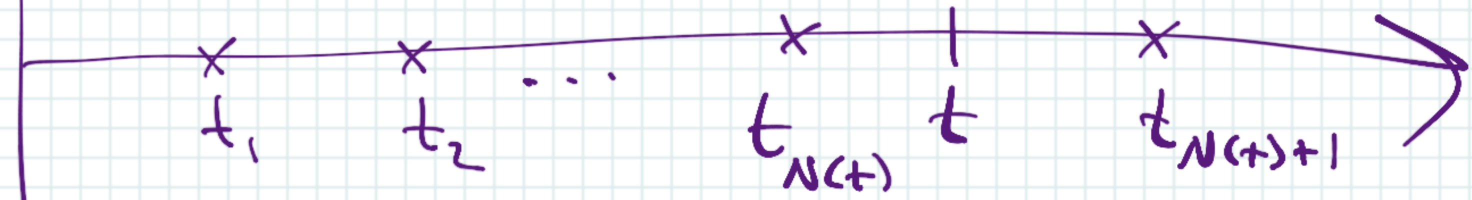
3) Generating Normal (μ, σ^2) rvs
Generating a Poisson rv

Inspection Paradox

for renewal processes:

$$\mathbb{P}(S(t) > x) \geq \mathbb{P}(X > x), \quad x \geq 0, \; t \geq 0$$

"Stochastically Larger"



$$S(t) = t_{N(t)+1} - t_{N(t)}$$
$$= \underline{X_{N(t)+1}}$$

$$\left(X_n = t_n - t_{n-1}, \quad n \geq 1 \right)$$

$$\Rightarrow \int_0^\infty \mathbb{P}(S(t) > x)\,dx \geq \int_0^\infty \mathbb{P}(X > x)\,dx$$

$$\| $$

$$\Rightarrow \quad E(S(t)) \geq E(X)$$

for <u>all</u> t !

Renewal Reward Thm

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds = \frac{E(R)}{E(X)} \quad \text{wp } 1$$

$$R = R_1 = \int_0^{X_1} S(s)\,ds = \int_0^{X_1} X_1\,ds$$

$$= X_1 \cdot X_1 = X_1^2$$

$$\boxed{\frac{E(R)}{E(X)} = \frac{E(X^2)}{E(X)}}$$

$$R_j = X_j^2 \quad , \; j \geq 1$$

CTMC $X(t)$

irreducible and pos. rec.

$$0 < E(T_{jj}) < \infty \qquad j \in \mathcal{S}$$

$$P_j = \frac{\frac{1}{q_j}}{E(T_{jj})} = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) = j\} ds$$

via Renewal Reward:

$$\left(\frac{E(R)}{E(X)} = \frac{\overbrace{E(H_j)}^{=\frac{1}{q_j}}}{E(T_{jj})} \right)$$

X_n = times between visits to state j

R_n = nth holding time in state j

$\left(E(R) = E(H_j) = \frac{1}{q_j} \right)$

Simulation

inverse transform method
for generating a rv X
with a given (desired)
CDF
$$F(x) = \mathbb{P}(X \le x), \quad x \in \mathbb{R}$$

If the inverse function is explicitely
known, $F^{-1}(y)$, $y \in (0,1)$ then simply
set $X = F^{-1}(U)$

1) What if F^{-1} is not known?

2) are there better algorithms than using F^{-1}?

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1) $N(\mu, \sigma^2)$ (Normal dist.) mean μ, variance σ^2

$Z \sim N(0,1)$ standard unit Normal

Z has density $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}$

$$\text{CDF} = F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \quad x \in \mathbb{R}$$

$$X = \sigma Z + \mu$$

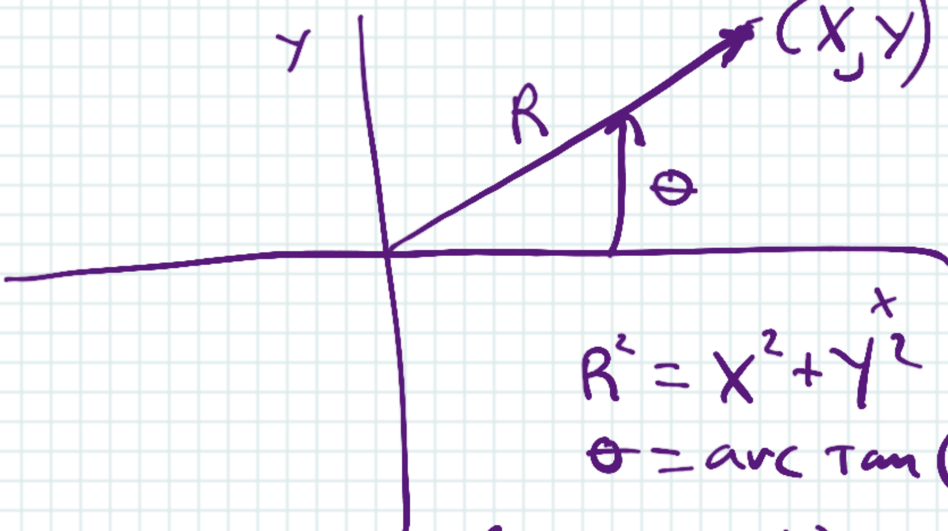
for any $\mu \in \mathbb{R}$ and $\sigma > 0$

$$\sim N(\mu, \sigma^2)$$

$\Rightarrow$ Sufficient to set an algorithm for simulating

$$Z \sim N(0,1)$$

Polar Method



Suppose X, Y are iid $N(0,1)$

$X = R\cos(\Theta)$
$Y = R\sin(\Theta)$

$R^2 = X^2 + Y^2$

$\Theta = \arctan\left(\frac{Y}{X}\right)$

Fact: $R^2 \sim \exp\left(\frac{1}{2}\right)$
$\Theta \sim \text{unif}(0, 2\pi)$

independent!

$g: \mathbb{R}^2 \mapsto \mathbb{R}^2$

(Proof based on Jacobian matrix determinant, of mapping $g(x,y) = \left(x^2+y^2, \arctan\left(\frac{y}{x}\right)\right)$ etc.)

Algorithm (Polar method)

1) Generate U_1, U_2

Set $R^2 = -2\ln(U_1) \sim \exp(\frac{1}{2})$

Set $\Theta = 2\pi U_2 \sim unif(0, 2\pi)$

Set $X = \sqrt{R^2}\cos(\Theta)$

$Y = \sqrt{R^2}\sin(\Theta)$

Then X, Y are iid $N(0,1)$ rvs.

Poisson dist.

$N \sim \mathrm{Poisson}(\alpha)$

$$\mathbb{P}(N=k) = e^{-\alpha} \frac{\alpha^k}{k!}, \quad k \geq 0$$

observation: $N \overset{\text{dist.}}{=} N(1)$

when $(N(t))$ is the counting process of a $PP(\alpha)$

($N(t) \sim \mathrm{Poisson}(\alpha t)$ for all $t \geq 0$. choose $t=1$)

(Thus it suffices to give an algorithm for simulating $N(1)$)

Algorithm:

$$\boxed{\text{Set } M = \min\{n \geq 1 : U_1 \cdot U_2 \cdots U_n < e^{-\alpha}\}}$$

$$= \min\{n \geq 1 : \ln(U_1 U_2 \cdots U_n) < -\alpha$$

$$= \min\{n \geq 1 : \underbrace{(-\tfrac{1}{\alpha}\ln(U_1)) + \cdots + (-\tfrac{1}{\alpha}\ln(U_n))}_{\text{iid interarrival times from } PP(\alpha)} > 1\}$$

recall: $-\frac{1}{\alpha}\ln(U) \sim \exp(\alpha)$

$PP(\alpha)$: (t_n) $\quad t_n = (-\frac{1}{\alpha}\ln(U_1)) + \cdots + (-\frac{1}{\alpha}\ln(U_n))$

$$= \min\{n \geq 1 : t_n > 1\} = N(1) + 1$$

$$N = N(1) = N(1)+1 - 1$$
$$= \overbrace{M - 1}$$

algorithm is thus:

Set $M = \min\{n \geq 1 : U_1 \cdots U_n < e^{-\alpha}\}$

Set $N = M - 1$.

then $N \sim \text{poisson}(\alpha)$.