IEOR 4106 lec 20

1) More on Simulation
2) Intro to Brownian Motion
Simulation (Inverse Transform method)

\[ F(x) = P(X \leq x), \quad x \in \mathbb{R} \]

\( F^{-1} \) is known

Generate \( U \) \( \sim \text{uniform}(0,1) \)

Set \( X = F^{-1}(U) \)
In the discrete case:

\[ p(k) = P(X = k), \quad k \geq 0 \]

pmf

Set

Generate \( U \)

\[ f(k) = P(X \leq k) \]

\[ X = \begin{cases} 
 0, & \text{if } U \leq p(0) \\
 1, & \text{if } p(0) < U \leq p(0) + p(1) = f(1) \\
 2, & \text{if } f(1) < U \leq f(2) \\
 \vdots \\
 k, & \text{if } f(k-1) < U \leq f(k) \\
 \end{cases} \quad k \geq 1 \]

\( (f(k) - f(k-1)) = p(k) \)
also have other methods both for continuous r.v.s and discrete r.v.s

Normal: Polar method

\[ P(X=k) = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi k!}} \]

Let \( W(\chi) \) be counting process for \( \rho(\chi) \)

\[ M = \min \{ n \geq 1 : U_1 \ldots U_n < \varepsilon^d \} \]

Set \( X = M^{-1} \)

\[ (M = N(1) + 1) \]

\[ N(1) \sim \text{Poisson} (d) \]
Simulating a reservoir model

\[ V_{n+1} = \left( V_n + S_n - c T_n \right)^+ \]

- \( V_n \): Water level right before \( n+1 \) rain storm
- \( t_n \): Time of \( n \text{th} \) rain storm
- \( T_n \): \( t_{n+1} - t_n \)
- \( S_n \): Amount of water of \( n \text{th} \) storm added to reservoir
Assume \( \{S_n\} \) iid
\[ G(x) = \Pr(S \leq x), \quad x \geq 0 \]
\( \{T_n\} \) iid
\[ A(x) = \Pr(T \leq x), \quad x \geq 0 \]
\( \{S_n\} \) and \( \{T_n\} \) are independent

\[ G^{-1} A^{-1} \]

Suppose we wish to simulate (for desired time \( N \))
\( (V_1, V_2, \ldots, V_N) \)
\[ V_{n+1} = (V_n + S_n - cT_n)^+ \]

\[ (V_0 = 0) \]

\[ 0 \leq n \leq N-1 \]

Generate \((U_{i,j}, Y_i)\), \((U_{2,j}, Y_2)\), \(\ldots\), \((U_{N,j}, Y_N)\) all iid \(\text{uniform}(g_1)\).

\[ V_1 = (V_0 + S_0 - t_0)^+ \]

\[ = (S_0 - t_0)^+ = (G^{-1}(U_1) - cA^{-1}(Y_1))^+ \]

\[ V_2 = (V_1 + S_1 - t_1)^+ = (V_1 + G^{-1}(U_2) - cA^{-1}(Y_2))^+ \]
\[ V_N = (V_{n-1} + S_{n-1} + \langle T_{n-1} \rangle) \]
\[ = (V_{n-1} + G^{-1}(V_n) - \langle A^{-1}(Y_n) \rangle) \]
\[
\checkmark
\]
Better to do sequentially:

- Enter \( n \)
- \( V_0 = 0 \)
- \( n = 0 \)

Step 1: output \((V_0, \ldots, V_n)\)

\[ n = (n + S^{-1}S)^T \]

\[ V_n = G^{-1}(u) \]

\[ T = A^{-1}(u) \]

\( \text{Generate} \]

\( \text{reset} n = n+1 \)

unless \( n < N \)

otherwise

Stop.
Relationship to Single-Server

\[ D_{n+1} = (D_n + S_n - T_n)^+ \]
\[ T_n = t_{n+1} - t_n \]

\[ W_n = D_n + S_n \]

\( D_n \) = delay in \( \text{line} \) of \( n^{th} \) customer

\( S_n \) = Sojourn time of \( n^{th} \) customer

\( t_n \)
$V(t) =$ water level at time $t \geq 0$

$= \text{work in queueing system at time } t$

$D_n = V(t_{n-}) \quad V(t_{n+}) = (V(t_{n-}) + S_n - T_n)^+$
Also "duality" with

Insurance Risk Business

claims i.i.d.

\[ R_t(x) = \text{reserve amount of business at } t \text{ given } R_0(x) = x \]

\[ \text{Ruin at time } t \]

\[ R_{t+4}(x) < 0 \]
Use simulation to estimate average delay:

\[ d \overset{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} D_i \]

\[ d \approx \frac{1}{N} \sum_{i=1}^{N} D_i \quad \text{for large } N \]

Can use simulation for the estimation of many quantities of interest.
**Brownian Motion**

Start with the Simple Symmetric RW

\[ \Pr(\Delta = 1) = \frac{1}{2} = \Pr(\Delta = -1) \]

\[ E(\Delta) = 0 \]

\[ \sigma^2 = \text{Var}(\Delta) = E(\Delta^2) - E(\Delta)^2 = E(\Delta^2) - 0 = E(\Delta) = 1 \]

\[ R_n = \sum_{i=1}^{n} \Delta_i, \quad i \geq 1 \]

\[ R_0 = 0 \]

\[ E(R_n) = 0 \quad \text{for} \quad n \geq 0 \]

\[ \text{Var}(R_n) = n \sigma^2 = n \]
Take an integer \( k \) large, fix \( t_{03} \)

\[
B_k(t) = \sum_{i=1}^{\Delta i \leq x} \frac{\Delta i}{\Delta k}
\]

\( E(B_k(t)) = 0 \)

Step sizes are now very small \( \pm \frac{1}{\Delta k} \)

\( t_k \) is large as \( t \) sets large

\( \{B_k(t) : t \geq 0\} \) is a new continuous time stochastic process
\[ \text{Var}(B_{k+1}) = \frac{1}{nk} \sum_{c=1}^{nk} \Delta_i \]

\[ B_{k+1} = \frac{1}{nk} \]

\[ E(B_{k+1}) = 0 \]

\[ \text{Var}(B_{k+1}) \rightarrow t \quad (\rightarrow t \text{ as } k \rightarrow \infty) \]

\[ E(R_{n_k}) = 0 \]

\[ \text{Var}(R_{n_k}) = 4 \]
\[ B_k(+) = \sqrt{t} \left( \frac{1}{\sqrt{t+k}} \sum_{i=1}^{[t+k]} \Delta_i \right) \]

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Delta_i \xrightarrow{\text{CLT}} N(0,1) \]

For each fixed \( t \),
\[ B_k(+) \xrightarrow{\text{CLT}} N(0, t) \]

\[ \begin{aligned} & \rho_\Delta = 1 \\ & \mu_\Delta = 0 \end{aligned} \]
\[ B_k(t+s) - B_k(s) = \frac{1}{\sqrt{k}} \sum_{i = sk+1}^{(k+1)s} \xi_i \leq \epsilon \]

In distribution only depends on the length \( t+s = t \)

Stationary increments

Two non-overlapping increments use different \( \xi_i \) hence are independent \( \Rightarrow \) independent increments
\[ \langle B_k (t) : t \geq 0 \rangle \xrightarrow{k \to \infty} \begin{cases} B(t) : t \geq 0 \end{cases} \]

Both stationary
& independent increments
( Normally distributed

\[ B(t) \sim \mathcal{N}(0, t) \]

\[ B(t+s) - B(s) \xrightarrow{\text{dis}} \mathcal{N}(0, t) \]

for all \( s \geq 0 \)
\( t \geq 0 \)
a stochastic process \( (B(t)) \)
is a standard BM if

1) \( B(0) = 0 \) and has continuous sample paths

2) \( (B(t)) \) has both stationary and independent increments

3) The increments are Normal
\[
B(s+t) - B(s) \overset{\text{dist}}{=} N(0, t)
\]
\( s \geq 0 \)
\( t > 0 \)
$t \geq 0, \quad \nu \in \mathbb{R}$

$X(t) = \sigma B(t) + \nu t, \quad t \geq 0$

is called a BM with drift $\nu$ and variance term $\sigma^2$

$E[X(t)] = E[\sigma B(t)] + \nu t$

$= 0 + \nu t = \nu t$

$\text{Var}(X(t)) = \sigma^2 \text{Var}(B(t))$

$= \sigma^2 t$
\[ Z \sim N(0,1) \]
\[ X = s \cdot Z + N \sim N(s, r^2) \]

\[ X(s+t) - X(s) \quad \overset{d}{=} \quad N(N^t, \sigma^2 t) \]

\[ B(s+t) - B(s) \quad \overset{d}{=} \quad N(0, t) \]
\((X(t))\) is a BM with 
\[ \text{drift } \mu, \text{ variance term } \sigma^2 \]
if
1) \(X(0) = 0\), continuous sample paths
2) Both stationary \& independent increments (Normal)
3) \(X(s+t) - X(s) \overset{d}{=} N(\mu t, \sigma^2 t) \)

\(t \geq 0, s \geq 0\)
\[ B(s+t) = B(s) + (B(s+t) - B(s)) \]

\[ N(s+t) = N(s) + (N(s+t) - N(s)) \]

\[ \text{(PP(\lambda) process)} \]

\[ \text{(counting process)} \]

\[ \Rightarrow \text{ Markov Property} \]