IEOR 4106 lec 20

1) More on Simulation

2) Intro to
   Brownian Motion

Simulation (Inverse Transform method)

$F(x) = \mathbb{P}(X \leq x), \quad x \in \mathbb{R}$

(F^{-1} is known)

Generate U (unif(0,1))

Set $X = F^{-1}(U)$ ✓

In the discrete case:

$$p(k) = \mathbb{P}(X = k), \quad k \geq 0$$

pmf

Generate U

set

$F(k) = \mathbb{P}(X \leq k)$

$$X = \begin{cases} 0, & \text{if } U \leq p(0) \\ 1, & \text{if } p(0) < U \leq p(0) + p(1) = F(1) \\ 2, & \text{if } F(1) < U \leq F(2) \\ \vdots & \\ k & \text{if } F(k-1) < U \leq F(k) \quad k \geq 1 \end{cases}$$

$(F(k) - F(k-1) = p(k)$ ✓

also have other methods both for Continuous rvs and discrete rvs

Normal : Polar method

Poisson(λ) :

$\mathbb{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0$

$M = \min\{n \geq 1 : U_1 \cdots U_n < e^{-\lambda}\} = N(1)+1$

set $X = M-1$

$(M = N(1)+1$

$N(1) \sim \text{Poisson}(\lambda))$

(Let $(N(t))$ be counting process for $PP(\lambda)$)

Simulating a reservoir model

$$V_{n+1} = \left(V_n + S_n - cT_n\right)^+$$

rate $-c$ / unit time

V_n = water level right before n^{th} rain storm

t_n = time of n^{th} rain storm

$T_n = t_{n+1} - t_n$

S_n = amount of water of n^{th} storm added to reservoir

Assume $\{S_n\}$ iid

CDF $G(x) = \mathbb{P}(S \le x)$, $x \ge 0$

(T_n) iid $A(x) = \mathbb{P}(T \le x)$, $x \ge 0$

$\{S_n\}$ $\{T_n\}$ are independent

G^{-1}, A^{-1}

Suppose we wish to simulate (for desired time N)

$(V_1, V_2, \ldots, V_N)$

$$\boxed{V_{n+1} = (V_n + S_n - cT_n)^+} \quad 0 \le n \le N-1$$

$$(V_0 = 0)$$

Generate $(U_1, Y_1)\ (U_2, Y_2)$

$\ldots\ldots\ (U_N, Y_N)$

all iid unif(0,1)

$$V_1 = (V_0 + S_0 - cT_0)^+$$

$$= (S_0 - T_0)^+ = (G^{-1}(U_1) - cA^{-1}(Y_1))^+$$

$$V_2 = (V_1 + S_1 - T_1)^+ = (V_1 + G^{-1}(U_2) - cA^{-1}(Y_2))^+$$

$$\ldots \quad V_N = \left(V_{N-1} + S_{N-1} + cT_{N-1}\right)^{+}$$

$$= \left(V_{N-1} + G^{-1}(U_N) - cA^{-1}(Y_N)\right)^{+}$$

$\checkmark$

Better to do Sequentially : Save storage space

Enter N, $V_0 = 0$, $n = 0$

1) while $n < N$, reset $n = n+1$

Generate U_1, U_2

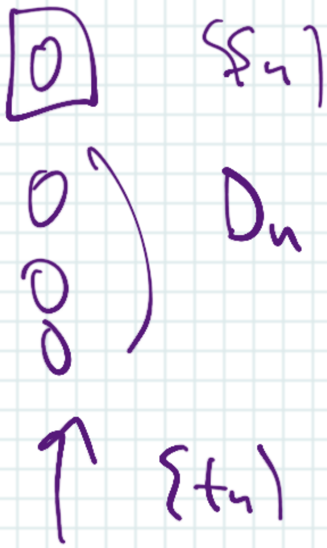
Set $S = G^{-1}(U_1)$, $T = A^{-1}(U_2)$

$$V_n = (V + S - T)^+$$

otherwise, $n = N$

Stop ; output $(V_1, \dots, V_N)$

Relationship to Single-Server FIFO Queue



$$\boxed{D_{n+1} = (D_n + S_n - T_n)^+, \; n \geq 0 \qquad T_n = t_{n+1} - t_n}$$

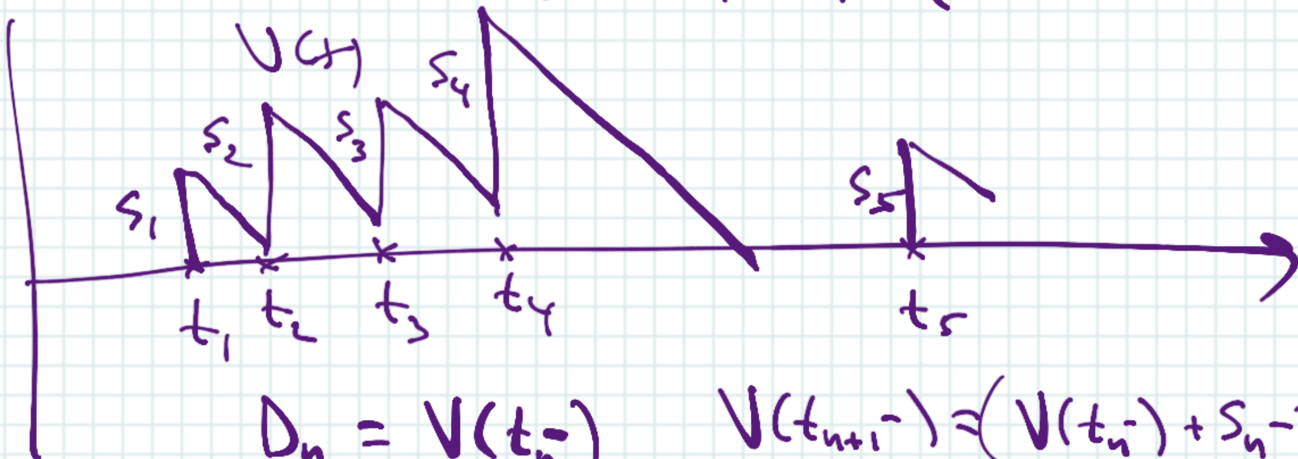
D_n = delay in line of n^{th} customer

$W_n = D_n + S_n$

= sojourn time of n^{th} customer

$V(t) =$ water level at time $t \geq 0$ $(c=1)$

$=$ work in Queueing System at time t



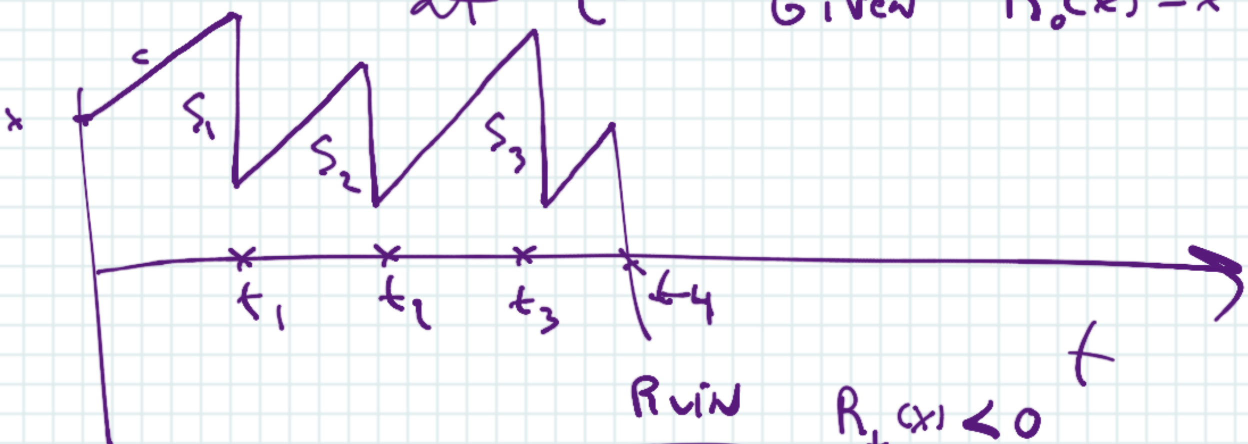
$D_n = V(t_n^-)$ $\quad V(t_{n+1}^-) = \left(V(t_n^-) + S_n - T_n\right)^+$

also "duality" with

Insurance Risk Business

earns \$ constant at rate c, claims iid (S_n)

$R_t(x)$ = reserve amount of Business at t Given $R_0(x) = x$



Ruin at time t_4 $R_{t_4}(x) < 0$

Use simulation to
estimate average delay:

$$d \overset{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} D_j$$

$$d \approx \frac{1}{N}\sum_{j=1}^{N} D_j \Leftarrow \quad \text{for } \underline{\underline{\text{large}}} \ N$$

Can use simulation for the
estimation of many quantities
of interest

Brownian Motion

Start with the Simple Symmetric RW $\quad R_n = \sum_{i=1}^{n} \Delta_i \,, \quad i \geq 1$

$R_0 = 0$

$\mathbb{P}(\Delta = 1) = \frac{1}{2} = \mathbb{P}(\Delta = -1)$

$E(\Delta) = 0$

$$\sigma^2 = Var(\Delta) = E(\Delta^2) - E^2(\Delta)$$
$$= E(\Delta^2) - 0$$
$$= E(1)$$
$$= 1$$

$E(R_n) = 0 \,, \quad n \geq 0$

$Var(R_n) = n\sigma^2 = n$

Take an integer k <u>large</u> fix it

$$B_k(t) \overset{def}{=} \sum_{i=1}^{[tk]} \frac{\Delta_i}{\sqrt{k}}$$

$[x]$ smallest integer $\leq x$

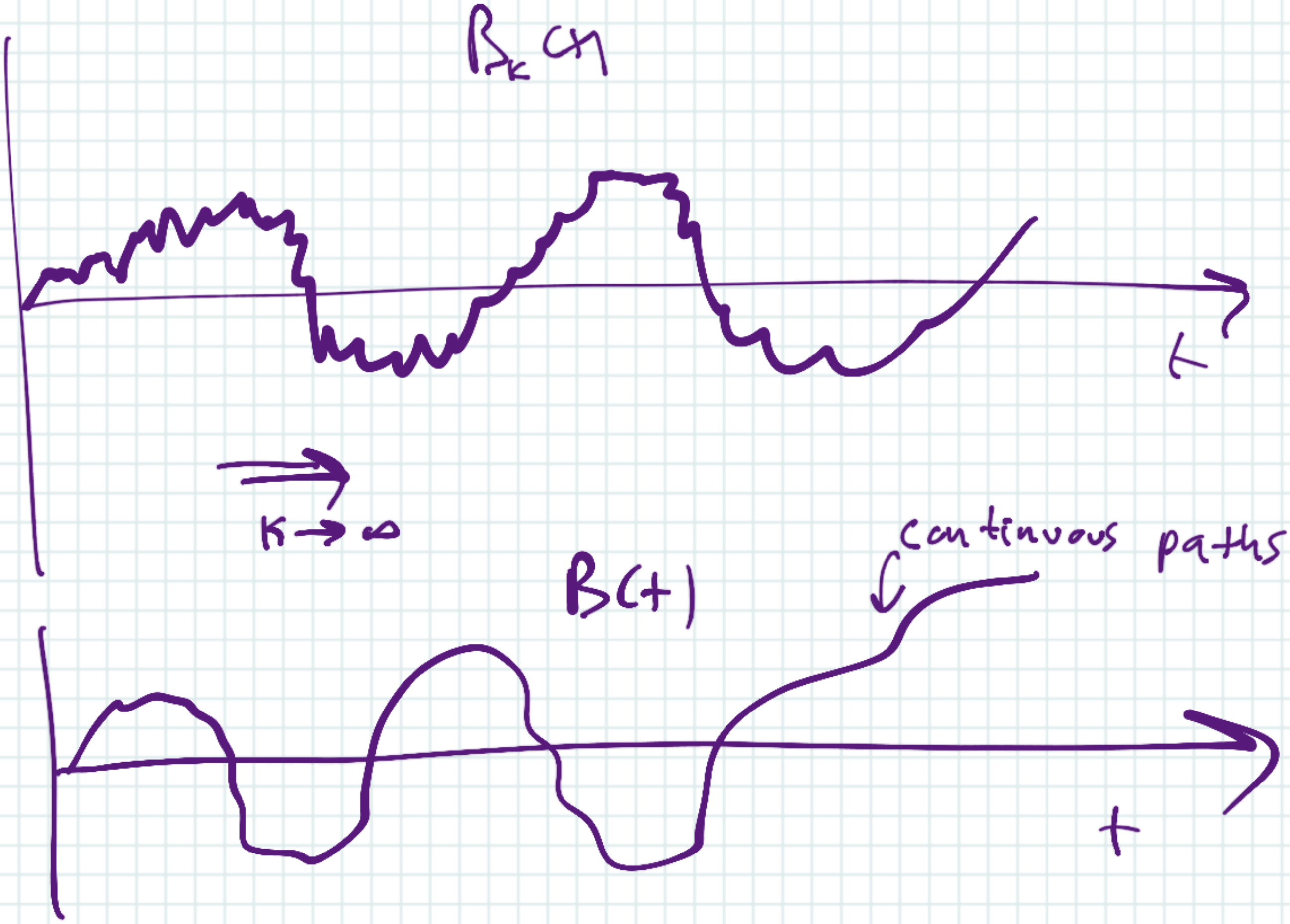
$(x-1 \leq [x] \leq x)$

$E(B_k(t)) = 0$

Step sizes are now very small $\pm \frac{1}{\sqrt{k}}$

tk is large as t gets large

$\{B_k(t) : t \geq 0\}$ is a new <u>continuous</u> time stoch. process



$B_K(t)$
t
$K \to \infty$
continuous paths
$B(t)$
t

$$\mathrm{Var}(B_K(t)) = \frac{\lfloor Kt \rfloor}{K} \xrightarrow[K\to\infty]{} t$$

$$\frac{(K-1)t}{K} \le \frac{\lfloor Kt \rfloor}{K} \le \frac{Kt}{K}$$

$$\downarrow \qquad\qquad \downarrow$$

$$t \qquad\qquad t$$

$$B_K(t) = \frac{1}{\sqrt{K}} \sum_{i=1}^{\lfloor tK \rfloor} \Delta_i$$

$$E(B_K(t)) = 0$$

$$\mathrm{Var}(B_K(t)) \approx t \quad (\to t \text{ as } K\to\infty)$$

$$E(R_n) = 0$$

$$\mathrm{Var}(R_n) = n$$

$$B_k(t) = \sqrt{t}\left(\frac{1}{\sqrt{tk}}\sum_{i=1}^{[tk]}\Delta_i\right)$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Delta_i \xRightarrow[\substack{CLT \\ n\to\infty}]{} N(0,1)$$

$$\begin{pmatrix} \sigma_\Delta^2 = 1 \\ \mu_\Delta = 0 \end{pmatrix}$$

For each fixed t,

$$B_k(t) \xrightarrow[CLT]{} N(0,t)$$

$$B_K(t+s) - B_K(s) \simeq \frac{1}{\sqrt{K}} \sum_{i=sK+1}^{(t+s)K} \Delta_i$$

in distribution, only depends on the length

$t+s-s = t$

Stationary increments

Two non-overlapping increments use different Δ_i hence are independent $\Rightarrow$ independent increments

Standard Brownian Motion (BM)

$$\{B_K(t) : t \geq 0\} \underset{K \to \infty}{\Longrightarrow} \{B(t) : t \geq 0\}$$

Both stationary & independent increments (Normally distributed)

$$B(t) \sim N(0, t)$$

$$B(t+s) - B(s) \overset{dis}{=} N(0, t)$$

for all $s \geq 0$, $t \geq 0$

a stochastic process $(B(t))$ is a standard BM if

1) $B(0)=0$, and has continuous sample paths

2) $(B(t))$ has both stationary and independent increments

3) the increments are Normal

$$B(s+t) - B(s) \overset{dist}{=} N(0, t)$$

$s \geq 0$

$t \geq 0$

$\sigma > 0$, $\mu \in \mathbb{R}$

$$X(t) = \sigma B(t) + \mu t, \quad t \geq 0$$

is called a BM with drift μ

and variance term σ^2

$$E(X(t)) = E(\sigma B(t)) + \mu t$$
$$= 0 + \mu t = \mu t$$

$$\mathrm{Var}(X(t)) = \sigma^2 \,\mathrm{Var}(B(t))$$
$$= \sigma^2 t$$

$$Z \sim N(0,1)$$

$$X = \sigma Z + \mu \sim N(\mu, \sigma^2)$$

$$X(s+t) - X(s)$$

$$\stackrel{d}{=} N(\mu t, \sigma^2 t)$$

$$B(s+t) - B(s) \stackrel{d}{=} N(0, t)$$

$(X(t))$ is a BM with drift μ, variance term σ^2 if

1) $X(0) = 0$, continuous sample paths

2) Both stationary & independent increments (Normal)

3) $X(s+t) - X(s) \stackrel{d}{=} N(\mu t, \sigma^2 t)$

$t \geq 0$
$s \geq 0$

$$B(s+t) = B(s) + \left(B(s+t) - B(s)\right)$$

PP(λ) counting process

$$N(s+t) = N(s) + \underline{\left(N(s+t) - N(s)\right)}$$

$\Rightarrow$ Markov Property !