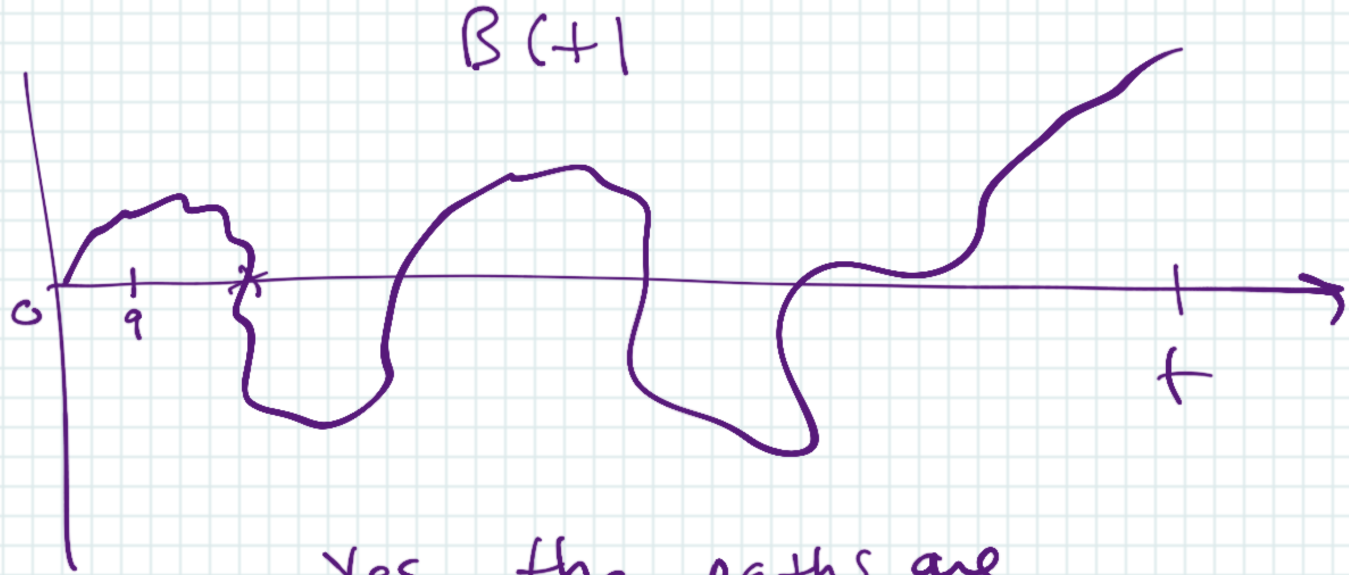
IEOR 4106 lec 21

More on Brownian Motion

$\{B(t)\}$ standard BM

1) $B(0)=0$, continuous paths
2) $(B(t))$ has both stationary & independent increments
3) increments are normally distributed

$$B(s+t)-B(s) \overset{d}{=} N(0,t)$$



Yes, the paths are continuous (wp1)

$\boxed{\int_0^t B(s)\,ds}$ is well defined

But: $B'(t)$ exists nowhere!

furthermore in any finite time interval $[0, t)$ the BM travels an <u>infinite</u> distance !

wp1

$$\cancel{\int_0^t dB(s)}$$

Riemann sense doesn't exist

a "new" integral

$$\int_0^t dB(s)$$

can be made rigorous using "stochastic calculus"

"Ito calculus"

$a > 0$

$$A = \{ t \in [0, a] : B(t) = 0 \}$$

$$\int_0^a I\{t : B(t) = 0\} \, \partial t$$

$$= \int_A \partial t \qquad = 0$$

A contains an uncountable number of values t but has length $= 0$

Review of Normal distribution

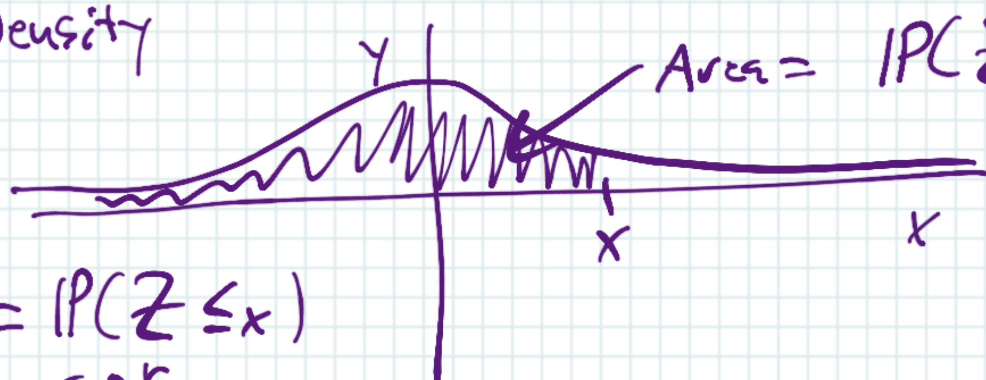
$Z \sim N(0,1)$

$E(Z) = 0$

$Var(Z) = EZ^2 - E^2(Z) = E(Z^2)$, where $E^2(Z) = 0$

density

$\Phi'(x) = \underset{\text{density}}{\underline{\phi(x)}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}$



Area $= \mathbb{P}(Z \le x) = \int_{-\infty}^{x} \phi(y)\,dy$

$\Phi(x) = \mathbb{P}(Z \le x)$
CDF

$$X \sim N(\mu, \sigma^2)$$

density

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R}$$

$$\boxed{X = \sigma Z + \mu}$$

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\left(Z \le \frac{x-\mu}{\sigma}\right)$$

CDF

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$F'(x) = \Phi'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \varphi\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R}$$

If $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ independent

then $X_1 + X_2 \sim N(\mu, \sigma^2)$ $\quad \mu = \mu_1 + \mu_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$

$(X(t))$ a BM with drift $\mu \in \mathbb{R}$

variance term σ^2

$(\sigma > 0)$

1) $X(0) = 0$, continuous sample paths

2) $(X(t))$ stationary & independent increments

3) $X(s+t) - X(s) \stackrel{dist}{=} N(\mu t, \sigma^2 t)$

$s \geq 0$

$t \geq 0$

$$\boxed{X(t) = \sigma B(t) + \mu t \quad t \geq 0}$$

Examples

1) A particle moves on a line according to a standard BM $(B(t))$. What is its expected position at time $t=6$? And its variance $t=6$?

$$E(B(6)) = 0, \quad Var(B(6)) = 6$$

2) Continuation

If its position $= 1.7$ at time $t=2$, what is its expected position at time $t=4$?

$$B(4) = B(2) + (B(4)-B(2)) = (B(2)-B(0)) + (B(4)-B(2))$$

$$\overset{dist.}{=} 1.7 + \underbrace{\sqrt{2}\,Z}_{N(0,2)}$$

independent, $N(0,2)$

$$E(B(4)) = 1.7 + E(\sqrt{2}\,Z)$$
$$= 1.7 + 0 = 1.7$$

3) The price of a commodity follows a BM

$\sigma^2 = 4$ $\qquad X(t) = \sigma B(t) + \mu t$

$\mu = -5$ $\qquad\qquad = 2B(t) - 5t \quad , \quad t \geq 0$

Given that $X(8) = 4$, what is $\boxed{\mathbb{P}(X(9) < 1)}$?

$$X(9) = X(8) + (X(9) - X(8))$$
$$\overset{dist.}{=} 4 + X(1) \qquad (X(1) \sim N(-5, 4) \sim 2Z - 5)$$
$$\overset{dist.}{=} 4 + 2Z - 5$$

$\mathbb{P}(X(9) < 1 \mid X(8) = 4)$

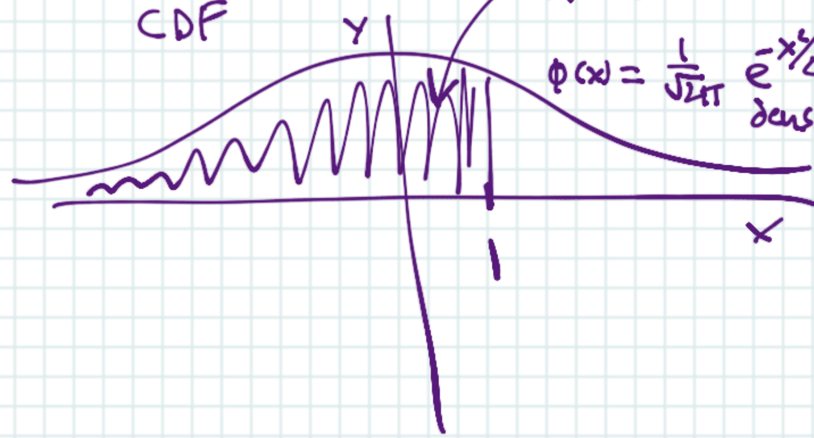
$= \mathbb{P}(2Z - 5 < -3)$

$= \mathbb{P}(Z < 1) = \underset{\text{CDF}}{\Phi}(1) = 0.8413$

Area to left of 1

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}$

density; "Bell curve"



Joint distributions of $(B(t))$

$0 < t_1 < t_2$ $\qquad$ Z_1, Z_2 iid $N(0,1)$

$$(B(t_1), B(t_2)) \overset{\text{distr}}{=} (\sqrt{t_1} Z_1, \sqrt{t_1} Z_1 + \sqrt{t_2 - t_1} Z_2)$$

$$B(t_2) = B(t_1) + (B(t_2) - B(t_1))$$

independent increments

$$B(t_1) = \sqrt{t_1} Z_1 \qquad B(t_2) - B(t_1) = \sqrt{t_2 - t_1} Z_2$$

$N(0, t_1)$, $\qquad$ $N(0, t_2 - t_1)$

$0 < t_1 < t_2 < \cdots < t_k$ $\qquad$ $Z_1, \cdots, Z_k$

$(B(t_1), \cdots, B(t_k))$ $\qquad$ iid $N(0,1)$

$B(t_1) = \sqrt{t_1}\, Z_1$

$B(t_2) = B(t_1) + \sqrt{t_2 - t_1}\, Z_2$

$\vdots$

$B(t_k) = B(t_{k-1}) + \sqrt{t_k - t_{k-1}}\, Z_k$

explicit construction of the desired random vector

$f(x_1, \cdots, x_k)$ Joint density of $(B(t_1), \cdots, B(t_k))$

$\{ B(t_1) = x_1, \cdots, B(t_k) = x_k \}$

$= \{ B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \cdots, B(t_k) - B(t_{k-1}) = x_k - x_{k-1} \}$

independent increment, $N(0, t_i - t_{i-1})$ $1 \le i \le k$

$$\boxed{f(x_1, \cdots, x_k) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2) \cdots f_{t_k - t_{k-1}}(x_k - x_{k-1})}$$

$f_t(x)$ denotes $N(0, t)$ density

$$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{t}}, \quad x \in \mathbb{R}$$

BM is a Markov Process

$$\underbrace{X(s+t)}_{\text{future}} = \underbrace{X(s)}_{\text{present state}} + \underbrace{(X(s+t) - X(s))}_{\text{independent of the past}}$$

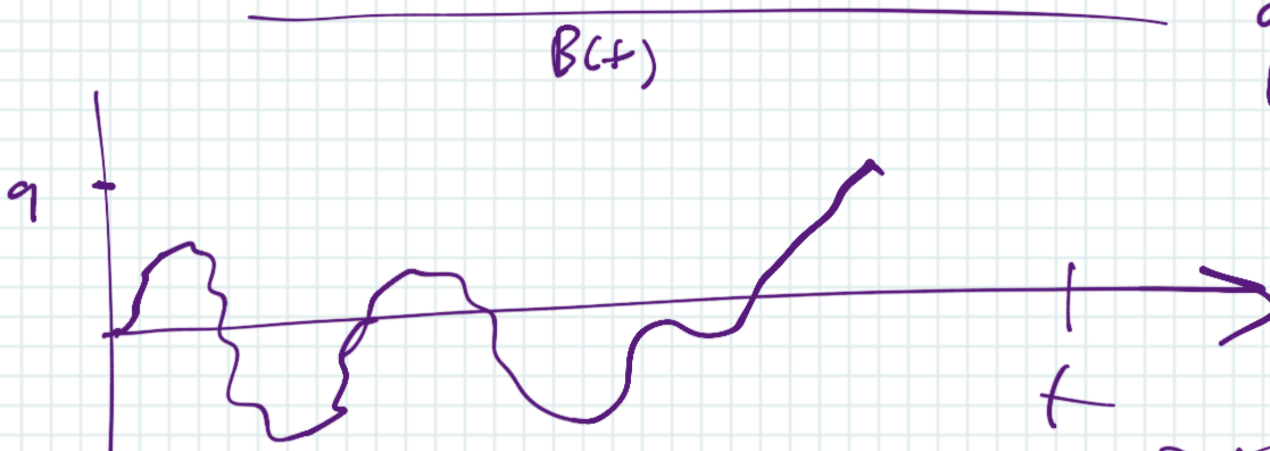
increment $(N(\mu t, \sigma^2 t))$

dist. of increment of length t

"Markov Property"

hitting probabilities

$a > 0$
$b > 0$



$$p(a) = \mathbb{P}\left(\{B(t)\} \text{ hits } a \text{ before } -b\right) \overset{?}{=} \boxed{\frac{b}{a+b}}$$

Recall $B_k(t) = \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} \Delta_i$

$\mathbb{P}(\{B_k(t)$ hits a before $-b\})$

$= \mathbb{P}(R_n^{(k)}$ hits a before $-b)$

$R_n^k = \frac{1}{\sqrt{k}} \sum_{i=1}^{n} \Delta_i$, $R_n = \sum_{i=1}^{n} \Delta_i$

$= \mathbb{P}(R_n$ hits $a\sqrt{k}$ before $b\sqrt{k})$

$= \frac{b\sqrt{k}}{a\sqrt{k} + b\sqrt{k}} = \frac{b}{a+b}$ ✓

$$\boxed{S(t) = S_0 e^{B(t)}} \quad t \geq 0$$

$S_0 > 0$

example of "geometric BM"

$$\ln(S(t)) = \ln(S_0) + B(t)$$

$S(t)$ has a <u>log normal</u> dist.

More generally

$$S(t) = S_0 \, e^{X(t)}$$

$$X(t) = \sigma B(t) + \nu t$$

as a model for the price of a risky asset over time

$S(t) = e^{B(t)}$ $\qquad S_0 = 1$

$\mathbb{P}\left((S(t)) \text{ hits } 2 \text{ before } \frac{1}{2} \right)$

$= \mathbb{P}(B(t) \text{ hits } \ln(2) \text{ before } -\ln(2))$

$a = \ln(2) = b$

$= \frac{1}{2}$ $\qquad S(t) = 5\, e^{B(t)}$ $\qquad S_0 = 5$

$\mathbb{P}((S(t)) \text{ hits } 6 \text{ before } 1)$

$= \mathbb{P}(\ln(5) + B(t) \text{ hits } \ln(6) \text{ before } 0)$

$= \mathbb{P}(B(t) \text{ hits } \ln(\frac{6}{5}) \text{ before } -\ln(5))$ $\qquad \begin{pmatrix} a = \ln(\frac{6}{5}) \\ b = \ln(5 \end{pmatrix}$

$$= \frac{b}{a+b} \quad \text{etc.}$$

if $X(t) = \sigma B(t) + \nu t$

$\nu \neq 0$

$\sigma \neq 1$

$p(a) = \mathbb{P}(\, (X(t)) \text{ hits } a \text{ before } -b\,)$

$$\boxed{p(a) = \frac{1 - e^{(2\nu/\sigma^2)b}}{e^{(-2\nu/\sigma^2)a} - e^{(2\nu/\sigma^2)b}}}$$

(Proof more difficult, uses Martingale Theory)

$$S(t) = S_0 e^{X(t)}$$

$$\mathbb{P}(S(t) \text{ hits } 2S_0 \text{ before } \tfrac{S_0}{2})$$

$$= \mathbb{P}(\ln(S_0) + X(t) \text{ hits } \ln(2) + \ln(S_0) \text{ before } \ln(S_0) - \ln(2))$$

$$= \mathbb{P}(X(t) \text{ hits } \ln(2) \text{ before } -\ln(2))$$

$$= p(a) \quad \text{where} \quad \begin{aligned} a &= \ln(2) \\ b &= \ln(2) \end{aligned}$$

$$E\left(S_0 e^{\sigma B(t)+\nu t}\right)$$

$X \sim N(\nu t, \sigma^2 t)$

$$= S_0 E\left(e^{\sigma B(t)+\nu t}\right) = S_0 E\left(e^{X}\right)$$

$$= S_0 e^{\bar{r} t}, \quad \bar{r} = \nu + \frac{\sigma^2}{2}$$

$t \geq 0$

Proved via use of moment generating functions MGF

MGF of X

$$M_X(1) = E\left(e^{X}\right), \quad \boxed{M_X(s) = E\left\{e^{sX}\right\} \quad s \in \mathbb{R}}$$