IEOR 4106 lec 22

More on BM:

Max, min rvs

$$\boxed{\begin{array}{l}\text{Geometric BM} \\ S(t) = S_0\, e^{X(t)}, \quad t \geq 0 \\ S_0 > 0\end{array}}$$

$X(t) = \sigma B(t) + \mu t, \quad t \geq 0$

Standard BM $(B(t))$

$a > 0$
$b > 0$

$$p(a) = \mathbb{P}(\{B(t)\} \text{ hits } a \text{ before } -b)$$
$$= \frac{b}{a+b}$$

$$\lim_{b\to\infty} p(a) = 1 \quad \text{for } \underline{\textbf{all}} \ a$$

$$\parallel$$

$$\mathbb{P}((B(t)) \text{ ever hits } a)$$
$$= \mathbb{P}(M \geq a), \quad M = \max_{t \geq 0} B(t)$$

$$\mathbb{P}(M = \infty) = 1$$

$$\mathbb{P}\big((B(t)) \text{ hits } -b \big) = \mathbb{P}(m \le -b)$$

$$= \lim_{a \to \infty} \boxed{\frac{a}{a+b}} = 1$$

$$m = \min_{t \geq 0} B(t)$$

$$\mathbb{P}(m = -\infty) = 1$$

$$\left(\begin{array}{l} \overline{\lim}\ B(t) = +\infty \\ \underline{\lim}\ B(t) = -\infty \end{array} \right)$$

Just like Simple Symmetric RW

$a > 0$

$$T_a = \min\{t > 0 : B(t) = a\}$$

$$\mathbb{P}(T_a < \infty) = 1 = \mathbb{P}(M \geq a)$$

$$E(T_a) = \infty$$

Can be shown that

$$\lim_{t \to \infty} \frac{B(t)}{t} = 0 \quad \text{wp1}$$

Simple Symetric RW

$$\left(\frac{R_n}{n} \xrightarrow[n \to \infty]{} E(\Delta) = 0 \right.$$

SLLN

For any n,

$$B(n) = B(1) + (B(2) - B(1)) + \cdots + B(n) - B(n-1)$$

$B(i) - B(i-1)$, $i \geq 0$

are iid $B(1) \sim N(0,1)$

stationary & independent increments

Let $Z_1, Z_2, \ldots\ldots Z_n$ be iid $\sim N(0,1)$

$$B(n) = \sum_{i=1}^{n} Z_i \quad \text{in distribution}$$

BM with drift $\nu \in \mathbb{R}$
variance term $\sigma > 0$

$X(t) = \sigma B(t)$ $\quad (\nu = 0)$

$\mathbb{P}((\sigma B(t))$ hits a before $-b)$

$= \mathbb{P}((B(t))$ hits $\frac{a}{\sigma}$ before $-\frac{b}{\sigma})$

$$= \frac{b/\sigma}{a/\sigma + b/\sigma} = \frac{b}{a+b}$$

similarly $\mathbb{P}(M = \infty) = 1$

$\mathbb{P}(m = -\infty) = 1$

$$\mu < 0$$

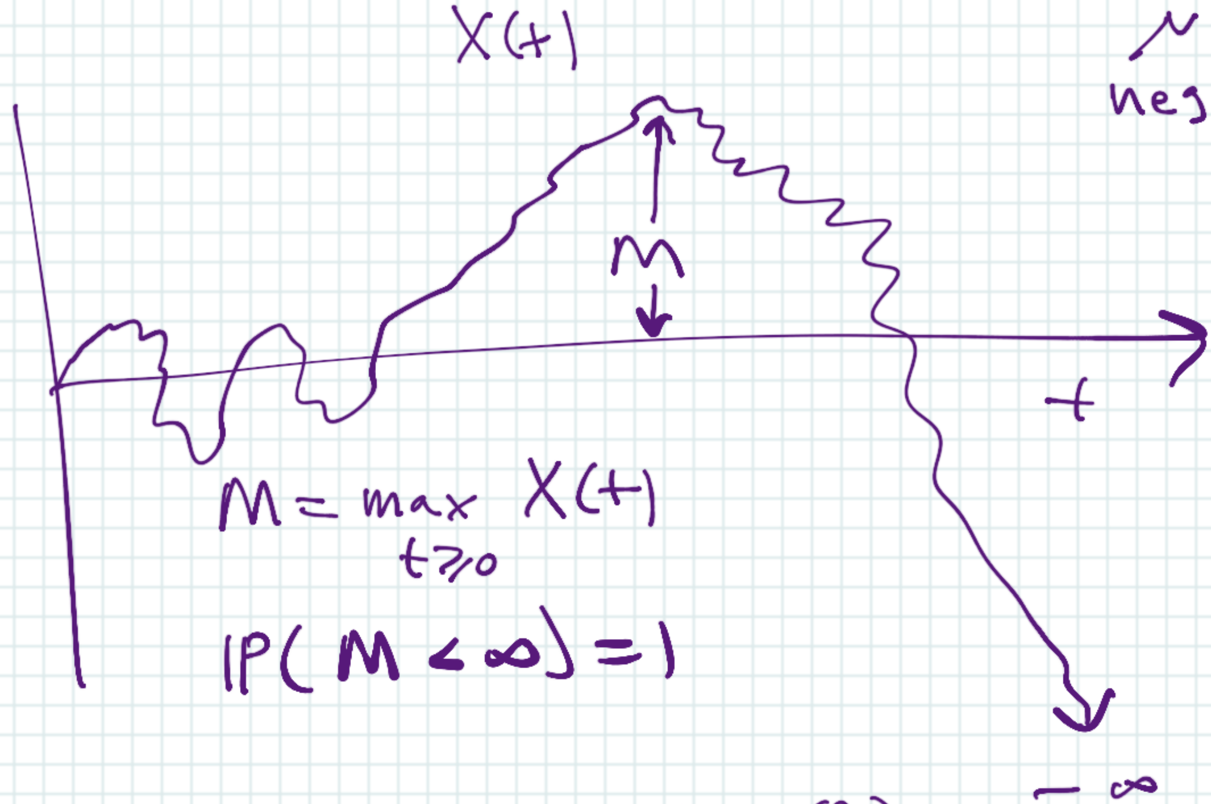
$$\frac{X(t)}{t} = \frac{\sigma B(t) + \mu t}{t}$$

$$\Downarrow \quad \left(\frac{\sigma B(t)}{t} \xrightarrow[t \to \infty]{} 0 \right)$$

$$\lim_{t \to \infty} \frac{\mu t}{t} = \mu < 0$$

$$\Rightarrow \quad X(t) \xrightarrow[t \to \infty]{} -\infty \quad \text{wp1}$$

$\mu < 0$
neg. drift



$$M = \max_{t \geq 0} X(t)$$

$$\mathbb{P}(M < \infty) = 1$$

Recall: for a simple RW (R_n), $p < \frac{1}{2}$

$$\left(\lim_{b \to \infty} p(a)\right) = \mathbb{P}(M \geq a) = \left(p/q\right)^a, \quad a \geq 0 \qquad \left(M = \max_{n \geq 0} R_n\right)$$

$\nu \neq 0$

$$p(a) = \frac{1 - e^{\left(\frac{2\nu}{\sigma^2}\right)b}}{e^{-\frac{2\nu}{\sigma^2}a} - e^{\left(\frac{2\nu}{\sigma^2}\right)b}}$$

$\nu < 0$

$$p(a) = \frac{1 - e^{-\frac{2|\nu|}{\sigma^2}b}}{e^{\frac{2|\nu|}{\sigma^2}a} - e^{-\frac{2|\nu|}{\sigma^2}b}}$$

$$\lim_{b\to\infty} p(a) = e^{-\alpha a} \quad a \geq 0, \qquad \boxed{\alpha \overset{\text{def}}{=} \frac{2|\nu|}{\sigma^2}}$$

$= \mathbb{P}(M \geq a)$ exp(α) tail

$$\nu > 0$$

$$Z \sim N(0,1)$$

$$-Z \sim N(0,1)$$

$$m = \min_{t \geq 0} (X(t))$$

$$\mathbb{P}(|m| \geq b) = e^{-\alpha b}, \quad b \geq 0$$

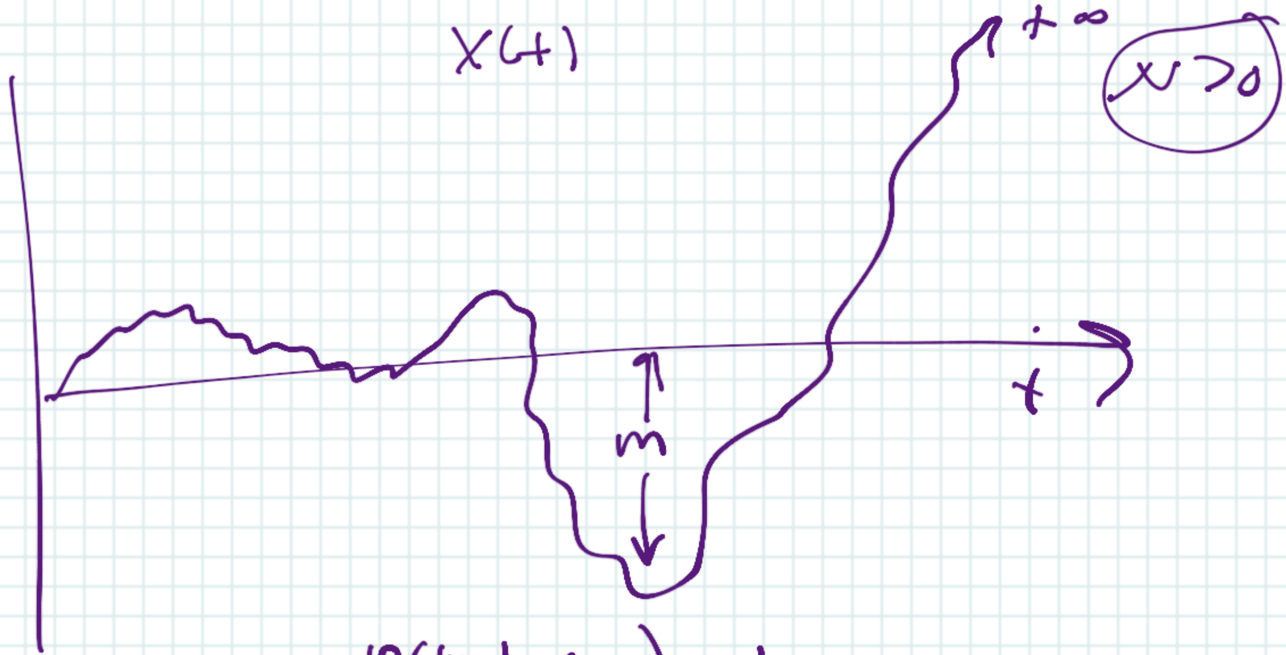
$$X(t) = \sigma B(t) + \nu t$$

$$\hat{X}(t) = -X(t) = -\sigma B(t) - \nu t, \quad t \geq 0$$

$$\left(\hat{X}(t)\right) \stackrel{dist}{=} \left\{\sigma B(t) - \nu t\right\}$$

BM with drift $-\nu$

Same σ



$$\mathbb{P}(|m| < \infty) = 1$$

$$\mathbb{P}(M = \infty) = 1$$

$$S(t) = 5\, e^{2B(t) - t} \qquad \begin{pmatrix} \sigma^2 = 4 \\ \mu = -1 \end{pmatrix}$$

$\mathbb{P}((S(t))$ even reaches as high as $10)$

$\mathbb{P}(\ln(5) + 2B(t) - t$ ever reaches (hits) $\ln(10))$

$= \mathbb{P}(2B(t) - t$ ever hits $\ln(\frac{10}{5}) = \ln(2))$

$$= \mathbb{P}(M \geq \ln(2)) = e^{-\frac{1}{2}\ln(2)} = e^{\ln(\frac{1}{\sqrt{2}})} = \boxed{\frac{1}{\sqrt{2}}}$$

$$\left(M = \max_{t \geq 0} \underline{2B(t) - t} \qquad \mathbb{P}(M \geq a) = e^{-\alpha a}, \quad \alpha = \frac{2|\mu|}{\sigma^2} = \frac{2}{4} = \frac{1}{2} \right)$$

$$S(t) = S_0 e^{X(t)} \qquad S_0 > 0$$

$$E(S(t)) = S_0 e^{\bar{r} t}, \quad t \geq 0$$

$$X(t) = \sigma B(t) + \nu t \qquad \bar{r} = \nu + \frac{\sigma^2}{2}$$

$$S(t) = 5 e^{2B(t) - t} \xrightarrow[t \to \infty]{} 0 \quad \text{wp1} \quad (\nu < 0)$$

$$E(S(t)) = 5 e^{t} \xrightarrow[t \to \infty]{} +\infty$$

$$\left.\begin{array}{l} \nu = -1 \\ \sigma^2 = 4 \end{array}\right\} \nu + \frac{\sigma^2}{2} = +1$$

$$E(S(t)) = S_0 E\left(e^{X(t)}\right) \qquad X := X(t)$$

$$X(t) \sim \boxed{N(\mu t, \sigma^2 t)}$$

Suppose $\boxed{X \sim N(\mu, \sigma^2)}$ $\qquad e^{s\mu + \frac{s^2\sigma^2}{2}} \quad s \in \mathbb{R}$

$$\boxed{M_X(s) \stackrel{def}{=} E\left(e^{sX}\right)} \quad // \quad s \in \mathbb{R}$$

(See Posted Lecture Notes on BM)

Moment Generating function of X

Note that if we have a formula for $M_X(s)$

$$M_X(1) = E\left(e^{X}\right)$$

$$\boxed{\begin{array}{l} \mu := \mu t \\ \sigma^2 := \sigma^2 t \end{array}}$$

$$X = \sigma Z + \mu$$

$$e^{sX} = e^{s\mu} \cdot e^{(s\sigma)Z}$$

$$E e^{sX} = e^{s\mu} \cdot E\left(e^{(s\sigma)Z}\right)$$

$$= e^{s\mu} M_Z(s\sigma)$$

$$M_Z(s) = e^{s^2/2}$$

$$s \in \mathbb{R}$$

$$E(S(t)) = S_0\, e^{\bar{r}t}, \quad t \geq 0$$

$$\boxed{\bar{r} = \mu + \frac{\sigma^2}{2}}$$

$$X_1(t) = \sigma_1 B_1(t) + \mu_1 t$$

independent

$$X_2(t) = \sigma_2 B_2(t) + \mu_2 t$$

$\{X_1(t) + X_2(t) : t \geq 0\}$

is a BM with $\mu = \mu_1 + \mu_2$

$\sigma^2 = \sigma_1^2 + \sigma_2^2$

$$\overset{dist}{=} \{\sigma B(t) + \mu t : t \geq 0\}$$

Similarly $\{X_1(t) - X_2(t) : t \geq 0\}$ $\quad \mu = \mu_1 - \mu_2$

$$\overset{dist}{=} \{\sigma B(t) - \mu t : t \geq 0\}$$

$$S_1(t) = 2e^{X_1(t)}$$

$$S_2(t) = 3e^{X_2(t)}$$

$(X_1(t), X_2(t))$ independent BMs

$$\mathbb{P}\big((S_1(t)) \text{ and } (S_2(t)) \text{ ever meet} \big)$$

$$\mathbb{P}\big((S_1(t) = S_2(t) \text{ some } t) \big)$$

$$\mathbb{P}\big(\ln(2) + X_1(t) = \ln(3) + X_2(t) \text{ ever} \big)$$

$$= \mathbb{P}\big(X_1(t) - X_2(t) = \ln(3) - \ln(2) = \ln(\tfrac{3}{2}) \big)$$

$$= \mathbb{P}\big(\sigma B(t) + \mu t \text{ ever hits } \ln(\tfrac{3}{2}) \big) \qquad \sigma^2 = \sigma_1^2 + \sigma_2^2$$

$$\mu = \mu_1 - \mu_2$$

$$= \mathbb{P}\big(M \geq \ln(\tfrac{3}{2}) \big)$$

if $\mu \geq 0$, then

$$\mathbb{P}(M = \infty) = 1$$

So $\mathbb{P}(M \geq \ln(\frac{3}{2})) = 1$

if $\mu < 0$

$$\alpha = \frac{2|\mu|}{\sigma^2}$$

$$\mathbb{P}(M \geq \ln(\tfrac{3}{2})) = e^{-\alpha \ln(\frac{3}{2})}$$

Variations of such questions:

$$\mathbb{P}\left(S_1(t) \geq 3 S_2(t) \quad \text{some } t \right)$$

$$= \mathbb{P}\left(\frac{S_1(t)}{S_2(t)} \geq 3 \right)$$

$$\frac{S_0(1)\, e^{X_1(t)}}{S_0(2)\, e^{X_2(t)}} = \frac{S_0(1)}{S_0(2)}\, e^{X_1(t) - X_2(t)}$$

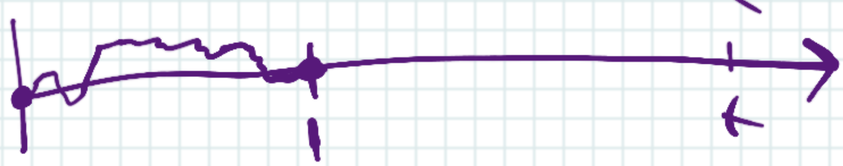
takes $\ln(\ )$ etc.

other processes related to BM:

integrated BM

1) $X(t) = \int_0^t B(s)\,ds, \quad t \geq 0$

2) Conditional on $B(0) = B(1) = 1$ what is the dist. of $\{B(t) : 0 \leq t \leq 1\}$



(Brownian Bridge)

$$S(t) = S_0 e^{X(t)}$$ for risky asset pricing

how about (better)

$$S_0 e^{X(t) + \sum_{i=1}^{N(t)} J_i}$$

← pp(λ)

← iid <u>Jumps</u>

(e.g. allow Jumps in price to occur)