4106 lec 3

Markov chains

Simple examples

Markov chains as recursions

Binomial Lattice model
for risky assets

Simple Random Walk

$R_0 = i \in \mathbb{Z} = \mathcal{S}$ $\qquad \boxed{0 < p < 1}$

iid $\{\Delta_n : n \geq 1\}$

$\mathbb{P}(\Delta = 1) = p$

$\mathbb{P}(\Delta = -1) = q = 1 - p$

$$R_n = i + \sum_{k=1}^{n} \Delta_k, \qquad n \geq 1$$

Gambler's Ruin MC (X_n) is a restricted RW

$$\mathcal{S} = \{0, 1, \dots, N\}$$

$$P_{i,i+1} = p \qquad 1 \leq i \leq N-1$$

$$P_{i,i-1} = q$$

$$P_{00} = 1 = P_{NN}$$

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with prob. $P_i(N)$

$$\lim_{n\to\infty} X_n = N$$

with prob. $1 - P_i(N)$

$$\lim_{n\to\infty} X_n = 0$$

$$(X_n) = \left(\begin{array}{l} \{ i, i+1, i-1, i+1, i+2, \dots, N, N, N, \dots \} \\ \{ i, i-1, i+1, i-1, i-2, \dots, 0, 0, 0, \dots \} \end{array} \right.$$

$$R_{n+1} = R_n + \underbrace{\Delta_{n+1}}_{\text{iid}}, \quad n \geq 0$$

Markov property is immediate

Examples

any iid sequence is a MC

(X_n) $\boxed{p(j) = \mathbb{P}(X = j) \quad j \in \mathcal{S}}$

$\mathbb{P}(X_{n+1} = j \mid X_n = i, \ldots)$

$= \mathbb{P}(X_{n+1} = j) = p(j)$

every row $i \in \mathcal{S}$ is identical

Product MC

$$X_n = X_0 \underbrace{Y_1 \cdot Y_2 \cdots Y_n}_{iid}$$

(X_0 is assumed independent of $\{Y_n\}$)

$$X_{n+1} = X_n \cdot \boxed{Y_{n+1}}$$

$\lim_{n\to\infty} X_n$ for a product MC?

$$\ln(X_n) = \ln(X_0) + \sum_{k=1}^{n} \ln(Y_k)$$

(when the $Y_k \geq 0$)

Random Walk

So if $E[\ln(Y)] > 0$, then

"pos. drift" so $\lim_{n\to\infty} \ln(X_n) \to \infty$ wp1

$$\Rightarrow e^{\ln(X_n)} = X_n \longrightarrow e^{\infty} = \infty \text{ wp1}$$

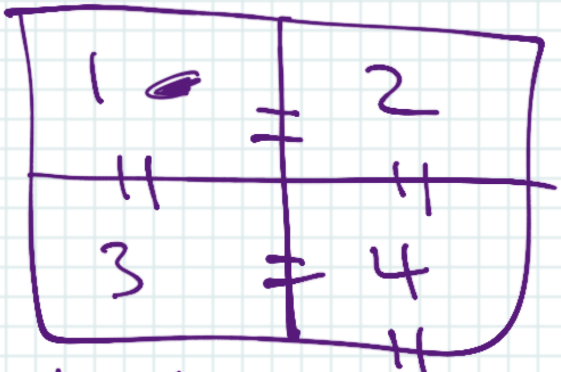
If $E \ln(Y) < 0$, (neg. drift)

then $\lim_{n\to\infty} \ln(X_n) = -\infty$ wp1

$$\Rightarrow X_n = e^{\ln(X_n)} \xrightarrow[n\to\infty]{} e^{-\infty} = 0 \text{ wp1}$$

Rat in an open maze

$S = \{0, 1, 2, 3, 4\}$



X_n = room visited after n^{th} move

"equally likely transition probabilities"

$$\mathbb{P}(X_{n+1} = 2 \mid X_n = 1) = \frac{1}{2} = \mathbb{P}(X_{n+1} = 3 \mid X_n = 1)$$

$$P_{12} = P_{13} = \frac{1}{2}, \quad P_{4,0} = P_{4,3} = P_{42} = \frac{1}{3}$$

etc, $P_{00} = 1$

$$P = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

$$\tau_{i,0} = \min\{n \geq 1 : X_n = 0 \mid X_0 = i\}$$

escape time when $X_0 = i$

$1 \leq i \leq 4$

$$E(\tau_{i,0}) = \; ?$$

(condition on X_1)

$$\boxed{E(\tau_{1,0}) = E(\tau_{1,0} \mid X_1 = 2)\,\mathbb{P}(X_1 = 2)}$$

$X_1 = 2$ or 3

$$\boxed{+\, E(\tau_{1,0} \mid X_1 = 3)\,\mathbb{P}(X_1 = 3)}$$

$$= \left((1 + E(\tau_{2,0}))\tfrac{1}{2} + (1 + E(\tau_{3,0}))\tfrac{1}{2} \right) = \left(1 + \tfrac{1}{2}\left(E(\tau_{2,0}) + E(\tau_{3,0})\right)\right)$$

$$E(T_{2,0}) = 1 + \frac{1}{2}E(T_{1,0}) + \frac{1}{2}E(T_{4,0})$$

$$E(T_{3,0}) = 1 + \frac{1}{2}E(T_{4,0}) + \frac{1}{2}E(T_{4,0})$$

$$E(T_{4,0}) = 1 + \frac{1}{3}E(T_{3,0}) + \frac{1}{3}E(T_{2,0})$$

a set of 4 linear equations, 4 unknowns

use linear algebra, Solution:

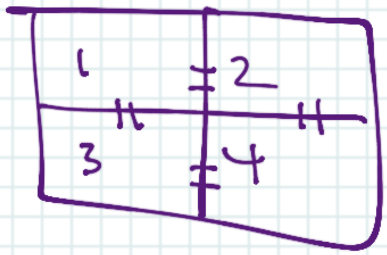
$$(13, 12, 12, 9)$$

$$\begin{matrix} 1 & 2 & 3 & 4 \end{matrix}$$

$$E(X) = E(X \mid Y=1)\,\mathbb{P}(Y=1) + E(X \mid Y=2)\,\mathbb{P}(Y=2)$$

$$\left(\text{if } \mathbb{P}(Y=1) + \mathbb{P}(Y=2) = 1\right)$$

Rat in closed maze



$S = \{1, 2, 3, 4\}$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$\pi_j \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} I\{X_n = j\}$$ = long-run proportion of time rat visits room j

$1 \le j \le 4$

$$\stackrel{?}{=} \frac{1}{4}$$

$\left(\begin{array}{c} X_0 = i \\ 1 \le i \le 4 \end{array}\right)$

Proved later, Yes!

$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ is a prob. dist. on $\mathcal{S}$

called the limiting or "stationary" dist. of the MC

Another
recursive example

Let $\{Y_n ; n \geq 0\}$ be any i.i.d. sequence values in $\mathbb{Z}$

$$\left(\begin{array}{l} p(a) = \mathbb{P}(Y = a) \quad a \in \mathbb{Z} \\ \sum_{a \in \mathbb{Z}} p(a) = 1 \end{array} \right)$$

$$M_n = \max\{Y_0, Y_1, \ldots, Y_n\}$$

observe:

$$M_{n+1} = \max\{Y_0, \dots, Y_{n+1}\}$$

$$= \boxed{\max\{M_n, Y_{n+1}\}}$$

Markov property immediate

$\mathbb{P}(M_{n+1} = j \mid M_n = i)$ only $\neq 0$ for $j \geq i$

$M_{n+1} \geq M_n \quad n \geq 0$

Non-decreasing

$$\mathbb{P}(M_{n+1}=i \mid M_n=i)$$
$$= \mathbb{P}(Y_{n+1} \leq i)$$
$$= \sum_{a \leq i} p(a)$$

$$\mathbb{P}(M_{n+1}=j \mid M_n=i)$$
$$j > i$$
$$= \mathbb{P}(Y_{n+1}=j) = p(j)$$

$\mathbb{P}(M_{n+1}=j \mid M_n=i) = 0$ for $j < i$

Markov chains as recursions

Suppose $f = f(i, u)$ is any function, and $(U_n : n \geq 0)$ is any iid sequence

and X_0 is independent of (U_n)

then $\boxed{X_{n+1} = f(X_n, U_n), n \geq 0}$ defines a MC

Simple random walk:

$$U_n = \Delta_{n+1}, \quad n \geq 0$$

$$f(i, u) = i + u$$

$$R_{n+1} = f(R_n, U_n), \quad n \geq 0$$

Product MC

$$X_n = X_0 Y_1 \cdots Y_n$$

$$U_n = Y_{n+1}, \quad n \geq 1$$

$$f(i, u) = i \cdot u$$

$$M_n = \max(Y_0, \ldots, Y_n)$$

$$f(i, v) = \max(i, v)$$

$$U_n = Y_n$$

Fact: Every MC (X_n)

can be expressed as a recursion. There exists a function $f = f(i, u)$ and an iid sequence (U_n) such that

$$X_{n+1} = f(X_n, U_n), \quad n \geq 0$$

The proof is based on how one can simulate rvs by using continuous

$\mathrm{Unif}(0,1)$ rvs U

$\boxed{\mathbb{P}(U \le x) = x, \quad x \in (0,1)}$

Suppose $F(x) = \mathbb{P}(X \le x)$, $x \in \mathbb{R}$

CDF

has an inverse function $F^{-1}(y)$, $y \in [0,1]$

$F \circ F^{-1}(y) = y$, $F^{-1} \circ F(x) = x \in \mathbb{R}$

Then, if U is Unif(0,1), then

$$X = F^{-1}(U)$$

is distributed as F

"Inverse Transform method"

Proof:

$$\mathbb{P}(F^{-1}(U) \leq x)$$

$$= \mathbb{P}(F \circ F^{-1}(U) \leq F(x))$$

$$= \mathbb{P}(U \leq F(x)) = F(x)$$

$$\left(\begin{array}{c} \mathbb{P}(U \leq a) = a \\ 0 \leq a \leq 1 \end{array} \right)$$

set $a = F(x)$

More generally define generalized inverse function

$$F^{-1}(y) = \min\{x : F(x) \geq y\}$$

$X = F^{-1}(U)$ still works

Example

$$\mathbb{P}(X=0) = 1-p$$

$$\mathbb{P}(X=1) = p \qquad \text{Bernoulli}(p)$$

Set $X = \begin{cases} 0 & \text{if } U \le 1-p \\ 1 & \text{if } U > 1-p \end{cases}$

$\mathbb{P}(X=0) = \mathbb{P}(U \le 1-p) = 1-p$ ✓

$\mathbb{P}(X=1) = \mathbb{P}(U > 1-p) = 1-(1-p) = p$ ✓

Suppose

$$\mathbb{P}(X=i) = p(i)$$

$$i = 1, 2, 3$$

Set

$$X = \begin{cases} 1 & \text{if } U \le p(1) \\ 2 & \text{if } p(1) < U \le p(1) + p(2) \\ 3 & \text{if } p(1)+p(2) < U \le p(1)+p(2)+p(3) = 1 \end{cases}$$

$$P(X=1) = \mathbb{P}(U \le p(1)) = p(1) \quad \text{etc}$$

it works!