4106 lec 4

1) More on Simulation of Markov Chains

2) n-step transition probabilities (Chapman-Kolmogorov Equations)

$$X_{n+1} = f(X_n, U_n), \quad n \geq 0$$

iid Sequence

$f = f(i, u)$

(X_0 is chosen independent of $\{U_n\}$)

always forms a MC ; converse

also true :

Every MC (X_n) can be expressed as a recursion: there exists a function $f = f(i, u)$ and an iid sequence (U_n) such that

$$X_{n+1} = f(X_n, U_n), \quad n \geq 0$$

In fact, (U_n) can be chosen as iid Unif$(0,1)$ rvs

CDF $\mathbb{P}(U \leq x) = x$, $x \in (0,1)$ (continuous Uniform distribution over $(0,1)$)

density $f(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & x \notin (0,1) \end{cases}$

$0 \leq a < b \leq 1$

$$\mathbb{P}(a < U \leq b) = b - a$$

Consider any CDF $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$ of a r.v.

If $F^{-1}(y)$ exists then $X = F^{-1}(U)$ has CDF F

Example

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

exponential dist.
at "rate" λ

$(X \geq 0)$

find $F^{-1}(y)$, $y \in (0,1)$

Solve $y = F(x)$ for x in terms of y

$y = 1 - e^{-\lambda x}$, $1 - y = e^{-\lambda x}$, $\ln(1-y) = -\lambda x$

$x = -\frac{1}{\lambda} \ln(1-y)$ ✓

$$x = F^{-1}(y) = -\frac{1}{\lambda}\ln(1-y)$$

$$X = \boxed{-\frac{1}{\lambda}\ln(1-U)} \overset{\text{dist.}}{=} \boxed{-\frac{1}{\lambda}\ln(U)}$$

because $1-U \overset{\text{dist}}{=} U \sim \text{unif}(0,1)$

$$\mathbb{P}(1-U \le x) = \mathbb{P}(U > 1-x) = 1-(1-x) = x \checkmark$$

$X = -\frac{1}{\lambda}\ln(U)$ has the $\exp(\lambda)$ dist.:

$$\mathbb{P}\left(-\frac{1}{\lambda}\ln(U) \le x\right)$$

$$= \mathbb{P}\left(\ln(U) > -\lambda x\right)$$

$$= \mathbb{P}\left(U > e^{-\lambda x}\right) = 1 - e^{-\lambda x} \quad \checkmark$$

"Inverse transform method"
for simulating copies X distributed as F

More generally define "generalized inverse function" of $F(x)$

$$F^{-1}(y) = \min \{x : F(x) \geq y\}$$

again $X = F^{-1}(U)$ yields a rv with CDF F (even if X is discrete)

Suppose X is discrete on $\{0, 1, 2, \ldots\}$

$$p(k) = \mathbb{P}(X = k) \quad k \geq 0$$

Generate U

Set $X = F^{-1}(U) = \begin{cases} 0 & \text{if } U \leq p(0) \\ 1 & \text{if } p(0) < U \leq p(0) + p(1) \\ \vdots & \vdots \\ k & \text{if } \sum_{j=0}^{k-1} p(j) < U \leq \sum_{j=0}^{k} p(j) \\ \vdots & \end{cases}$

$p(k) = \mathbb{P}(X = k)$ ✓

define $f(i, u) \overset{def}{=} F_i^{-1}(u)$ $\quad U_0, U_1$ $\quad i \in \mathcal{S}.$

F_i = CDF of i^{th} row of P $\quad i \in \mathcal{S}$

F_i^{-1} = generalized inverse function of F_i

if $X_n = i$, then

$X_{n+1} = f(i, U_n) = F_i^{-1}(U_n)$ ✓

When F^{-1} does not have a closed form then we can't use this $\boxed{X = F^{-1}(U) \text{ method}}$.

Example: X has a $N(0,1)$ dist. (Normal)

$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$. $F^{-1}(y) = ?$

$$R_{n+1} = R_n + \Delta_{n+1}$$

$$U_n \stackrel{\text{def}}{=} \Delta_{n+1}$$

$$f(i, u) = i + u$$

✓

n-step transition probabilities

$$P_{i,j}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) \quad i,j \in \mathcal{S}$$

$$\boxed{n \geq 1}$$

$$\boxed{P^{(n)} = \left(P_{ij}^{(n)}\right)_{i,j \in \mathcal{S}}} \qquad P^{(1)} = P$$

$$\left(P^{(0)} \stackrel{\text{def}}{=} I\right)$$

identity matrix

n^{th}-step transition matrix

Prop:

$$P^{(n)} = P^n = \underbrace{P \times P \times \cdots \times P}_{n \text{ times}}$$

Proof based on Chapman Kolmogorov Equations:

$$P_{ij}^{(n+m)} = \sum_{k \in S} P_{ik}^{(n)} \cdot P_{kj}^{(m)}$$

$n \geq 0$
$m \geq 0$
$i, j \in S$

via Markov Property

$$P^{(2)}_{ij} =$$

$$\boxed{n = m = 1}$$

$$P^{(1)} = P$$

$$P^{(1+1)}_{ij} = \sum_{k \in \mathcal{S}} P_{ik} P_{kj} \quad \text{for all } i,j \in \mathcal{S}$$

matrix multiplication!

$$\Rightarrow P^{(2)} = P \times P = P^2$$

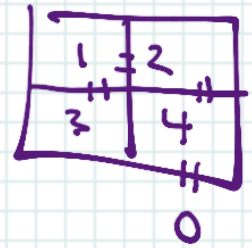
$$P^{(3)}_{ij} = P^{(2+1)}_{ij} = \sum_{k \in \mathcal{S}} P^{(2)}_{ik} P_{kj} = \sum_{k \in \mathcal{S}} P^{2}_{ik} P_{kj} \Rightarrow$$

$$P^{(3)} = P^2 \times P = P^3$$

Complete by induction on n

$$P^{(n+1)} = P^{(n)} \cdot P = P^n \cdot P = P^{n+1}$$

Examples Rat in open maze



$\mathcal{S} = \{0, 1, 2, 3, 4\}$

$$P^{(2)}_{1,1} = \mathbb{P}(X_2 = 1 \mid X_0 = 1) = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \frac{1}{2}$$

$$P^2 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 2 & \frac{1}{6} & 0 & \frac{5}{12} & \frac{5}{12} & 0 \\ 3 & \frac{1}{6} & 0 & \frac{5}{12} & \frac{5}{12} & 0 \\ 4 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{array}$$

$\mathbb{P}(X_2 = 2 \mid X_0 = 2)$

$$\mathbb{P}\left(\text{Rat escapes } \underline{\text{No later}} \text{ than } n=7 \,\middle|\, X_0=1\right)$$

$$= P^{(7)}_{1,0}$$

$P_{00}=1$

$\mathbb{P}(X_{n+1}=0 \mid X_n=0)$

$$= \mathbb{P}\left(\text{Rat escapes } \underline{\text{by}} \text{ time } 7 \,\middle|\, X_0=1\right)$$

Gambler's Ruin MC

$$\mathcal{S} = \{0, 1, \dots, N\}$$

$$P_{00} = 1 = P_{NN}$$

$$\mathbb{P}(\text{Gambler \underline{stops by} time } n \mid X_0 = i)$$

$$1 \le i \le N-1$$

$$= \boxed{P^{(n)}_{i,0} + P^{(n)}_{i,N}}$$

Rat in closed maze

$$\mathcal{S} = \{1, 2, 3, 4\}$$

$$P^{(n)}_{i,j} = \mathbb{P}(X_n = j \mid X_0 = i)$$

$1 \leq i, j \leq 4$

Weather modeling

$$W_n = \begin{cases} 1 & \text{if it rains on } n\text{th day} \\ 0 & \text{if no rain on } n\text{th day} \end{cases}$$

$$S = \{0, 1\}$$

Assume a MC

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{cc} 0 & 1 \\ \left[\begin{array}{cc} .7 & .3 \\ .4 & .6 \end{array}\right] \end{array}$$

(clearly not a realistic model)

Instead, assume that

$$X_n = (W_{n-1}, W_n) \text{ is a MC}$$

$$\mathcal{S} = \left\{ \underset{\underset{0}{\shortparallel}}{(0,0)}, \underset{\underset{1}{\shortparallel}}{(0,1)}, \underset{\underset{2}{\shortparallel}}{(1,0)}, \underset{\underset{3}{\shortparallel}}{(1,1)} \right\}$$

more realistic.

Suppose that

$0 = (0,0) \xrightarrow{.8} (0,0) = 0$

$ \xrightarrow{.2} (0,1) = 1$

$1 = (0,1) \xrightarrow{.5} (1,0) = 2$

$ \xrightarrow{.5} (1,1) = 3$

$2 = (1,0) \xrightarrow{.6} (0,0) = 0$

$ \xrightarrow{.4} (0,1) = 1$

$3 = (1,1) \xrightarrow{.3} (1,0) = 2$

$ \xrightarrow{.7} (1,1) = 3$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{bmatrix} \end{array}$$

$\mathbb{P}$(Rainy on Wed | No rain on Sunday and no rain on Monday) = ?

$$\mathbb{P}\left(X_{n+2} = \begin{matrix} (0,1) = 1 \\ \text{or} \\ (1,1) = 3 \end{matrix} \;\middle|\; X_n = (0,0)\right) = P^{(2)}_{0,1} + P^{(2)}_{0,3}$$

$$P^2 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

$$P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .64 & .16 & .10 & .10 \\ .30 & .20 & .15 & .35 \\ .48 & .12 & .20 & .20 \\ .18 & .12 & .21 & .49 \end{bmatrix} \end{array}$$

$$P^{(2)}_{0,1} + P^{(2)}_{0,3} = .16 + .10 = .26$$

Even Better:

$X_n = (W_{n-2}, W_{n-1}, W_n)$

& has $\underline{\underline{8}}$ states

etc.

other MC examples

Simple Inventory model

$$\mathcal{S} = \{0, 1, \dots, 10\}$$

$\boxed{X_0 = 10}$

headphones (pairs) for sale

$X_n =$ inventory at end of n^{th} week

$D_n =$ demand on the n^{th} week

(D_n) iid poisson(α)

$\boxed{E(D) = \alpha = Var(D)}$ $\mathbb{P}(D = j) = e^{-\alpha} \frac{\alpha^j}{j!}, \quad j \geq 0$

If $X_n > 0$, then

$$\underline{X_{n+1}} = (X_n - D_{n+1})^+$$

$$= \max\{X_n - D_{n+1}, 0\}$$

If $X_n = 0$, then <u>restock back to 10</u>

$$X_{n+1} = (10 - D_{n+1})^+$$

MC since it <u>is</u> a recursion

$$P = (P_{ij}) \text{ for } \mathcal{S} = \{0, 1, 2, 3\}$$

$$X_0 = 3$$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \mathbb{P}(D \geq 3) & \mathbb{P}(D=2) & \mathbb{P}(D=1) & \mathbb{P}(D=0) \\ 1 & \mathbb{P}(D \geq 1) & \mathbb{P}(D=0) & 0 & 0 \\ 2 & \mathbb{P}(D \geq 2) & \mathbb{P}(D=1) & \mathbb{P}(D=0) & 0 \\ 3 & \mathbb{P}(D \geq 3) & \mathbb{P}(D=2) & \mathbb{P}(D=1) & \mathbb{P}(D=0) \end{array}$$