IEOR 4106 lec 5

1) More examples of Markov chains including the Binomial Lattice model for risk assets

2) communication classes, recurrence, transience

Reservoir model

X_n = water level at the end of the n^{th} month

$$X_{n+1} = \left(X_n + S_n - T_n\right)^+$$

S_n: rain during $n+1$ month

T_n: water (potential) used during $(n+1)^{st}$ month

$x^+ = \max\{0, x\}$ positive part of x

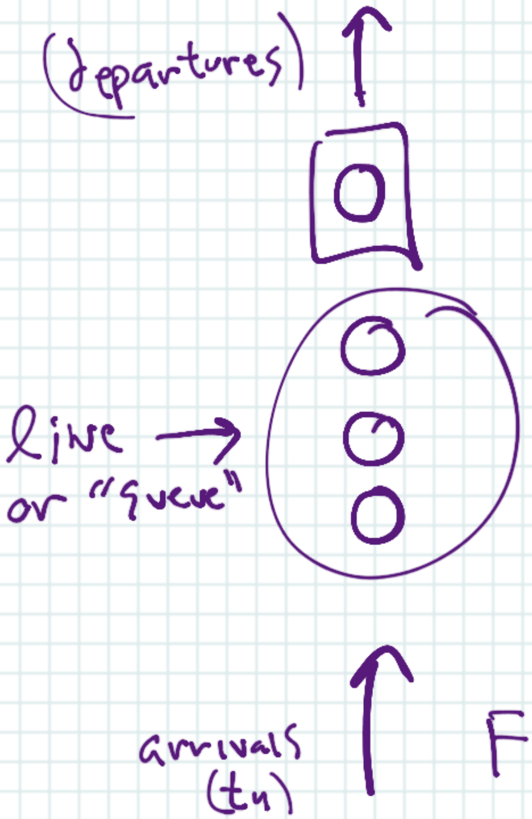
To ensure it forms a MC, we can assume

$\left(U_n = S_n - T_n\right)$ $\{S_n - T_n : n \geq 0\}$ is iid

(for example assume that each of $\{S_n\}$ and $\{T_n\}$ is iid)

ATM Machine

(queue) (single-server model FIFO)



FIFO

S_n = n^{th} customer's service time

t_n = arrival time of n^{th} customer

$T_n = t_{n+1} - t_n$ n^{th} interarrival time

D_n = delay in the queue (line) of n^{th} customer

$$D_{n+1} = \left(D_n + S_n - T_n \right)^+$$

$T_n = t_{n+1} - t_n$

$\square$

if (S_n) and (T_n) are iid, this queueing model is denoted by "GI|GI|1" queue

Transient / recurrent States

$i \in \mathcal{S}$

for a MC (X_n)

$$T_{i,i} = \begin{cases} \min\{n \geq 1 : X_n = i \mid X_0 = i) \\ \infty \quad \text{if no return to } i \end{cases}$$

"return time to state i"

$f_i = \mathbb{P}(T_{ii} < \infty)$ = Prob. that the chain returns

If $f_i = 1$, then state i is called recurrent

If $f_i < 1$, then state i is called transient

The Simple Random Walk

$0 < p < 1$

$p \neq \frac{1}{2}$ all States are transient

$p = \frac{1}{2}$ all States are recurrent

$$\boxed{N_i = \sum_{n=0}^{\infty} I\{X_n = i \mid X_0 = i\} \geq 1}$$

= total # visits to state i over all time $\mid X_0 = i$

either $\mathbb{P}(N_i < \infty) = 1$ (transient),
or $\mathbb{P}(N_i = \infty) = 1$ (recurrent)

$$\mathbb{P}(N_i = k) = f_i^{k-1}(1-f_i), \quad k \geq 1$$

Geometric dist. with "success prob" $= 1-f_i$

$$E(N_i) = \frac{1}{1-f_i}$$

Observe thus that

State i is recurrent $(f_i = 1)$

if and only if

$$\boxed{E(N_i) = \infty} \qquad \left(= \frac{1}{1-f_i} \right)$$

transient $(f_i < 1)$ otherwise ; $E(N_i) < \infty$.

$$E\left[I\{X_n = i \mid X_0 = i\} \right] = \mathbb{P}(X_n = i \mid X_0 = i)$$

$$= p_{i,i}^{(n)}$$

$$E(N_i) = \sum_{n=0}^{\infty} P_{i,i}^{(n)}$$

example: Simple Random Walk

$0 < p < 1$

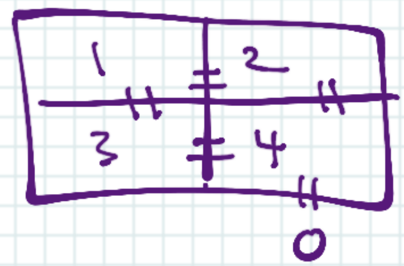
$$E(N_i) = \sum_{n=0}^{\infty} \binom{2n}{n} p^n (1-p)^n \qquad \binom{2n}{n} = \frac{(2n)!}{n!\,n!}$$

one can show that

$$\boxed{\begin{array}{ll} E(N_i) < \infty & p \neq \frac{1}{2} \\ E(N_i) = \infty & p = \frac{1}{2} \end{array}}$$

Examples:

Rat in open maze

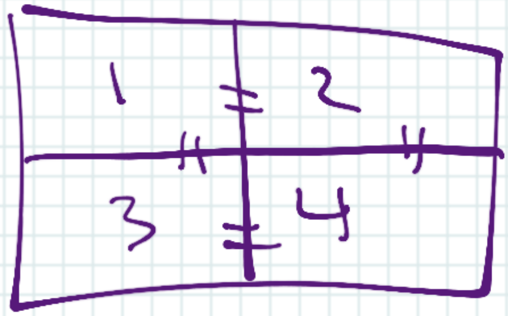


$S = \{0, 1, 2, 3, 4\}$

0 is recurrent

each of 1, 2, 3, 4 is transient

Rat in closed maze



$\mathcal{S} = \{1, 2, 3, 4\}$

all States must be recurrent

$$\pi_j = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} I\{X_n = j \mid X_0 = i\}$$

= long-run proportion of time rat spends in Room j

Should intuitively hold that

$$\pi_j = \frac{1}{4} \quad \text{wp } 1$$

$$1 \le j \le 4$$

$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ then forms a prob. distribution

called the limiting or stationary dist. of the MC.

Another Markov model
Binomial Lattice Model

$$S_{n+1} = S_n \cdot Y_{n+1}, \quad n \geq 0$$

S_n = price/share of a stock at end of nth day

$(S_0 > 0)$

$\{Y_n\}$ is iid

$\mathbb{P}(Y = u) = p$ $\quad (0 < p < 1)$

$\mathbb{P}(Y = d) = 1 - p$

$\boxed{0 < d < 1 + r < u}$; r = interest rate

$x \longmapsto (1+r)^n x$ at day n

$$
\begin{tikzcd}[column sep=large]
 & n=1 & n=2 \\
 & uS_0 \arrow[r, "p"] \arrow[dr, "1-p"'] & u^2 S_0 \\
S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & & duS_0 \\
 & dS_0 \arrow[r, "p"] \arrow[dr, "1-p"'] & udS_0 \\
 & & d^2 S_0
\end{tikzcd}
$$

$$S_n = S_0 Y_1 \cdot Y_2 \cdots Y_n$$

$$\mathbb{P}\left(S_n = S_0 u^i d^{n-i}\right) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$0 \le i \le n$$

Binomial (n, p) dist

$$\mathcal{S} = \left\{ S_0 u^i d^j \;:\; i \ge 0,\ j \ge 0 \right\}$$ "lattice of points"

$$E(Y) = pu + (1-p)d$$

$$E(S_n) = E(S_0 Y_1 \cdots Y_n)$$
$$= S_0 E(Y_1 \cdots Y_n)$$
$$= S_0 (E(Y))^n$$
$$= S_0 [pu + (1-p)d]^n, \quad n \geq 0$$

$$S_n = S_0 Y_1 \cdots Y_n \xrightarrow[n \to \infty]{} ?$$

$$\frac{\ln(S_n)}{n} = \ln(S_0)/n + \boxed{\frac{\sum_{m=1}^{n} \ln(Y_m)}{n}} = \frac{R_n}{n}$$

Random walk with

(ii) increments $\Delta_m = \ln(Y_m)$

If $E(\Delta) > 0$, then $R_n \xrightarrow[n\to\infty]{} +\infty$ wp1 $\Longrightarrow$

$$S_n = S_0 e^{R_n} \xrightarrow[n\to\infty]{} S_0 e^{\infty} = \infty$$

If $E(\Delta) < 0$, then

$R_n \to -\infty$ wp1

so $S_n = S_0 e^{R_n} \longrightarrow S_0 e^{-\infty} = 0$

wp1

p, u, d

Option Pricing under the BLM

at time $\boxed{T} > 0$

you get payoff

$$C_T = \boxed{(S_T - K)^+}$$

European call option

$K > 0$ constant "strike price"

(S_u)

What is a fair price for such an option?

$$C_T = \left(\frac{1}{T} \sum_{n=1}^{T} S_n \quad - K \right)^+$$

asian call option

$$C_T = (K - S_T)^+$$

"put" option

Communication classes for a MC (X_n), $\mathcal{S}$

We say that j is reachable from i $\quad i,j \in \mathcal{S}$

if for some n

$$p^{(n)}_{i,j} = \mathbb{P}(X_n = j \mid X_0 = i) > 0$$

$$\boxed{i \longmapsto j}$$

if $i \longmapsto j$ and

$j \longmapsto i$

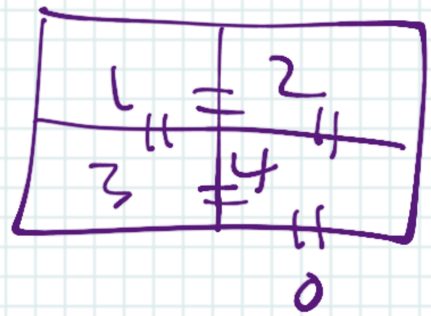
then we write $\boxed{i \longleftrightarrow j}$

and say i and j communicate

$P^{(n)}_{ij} > 0$ some n

$P^{(m)}_{ji} > 0$ some m

Example Rat in open maze



0 only communicates with itself

but $\{1, 2, 3, 4\}$ all communicate

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when all states in $\mathcal{S}$ communicate we call the chain <u>irreducible</u>.

1) every state communicates with itself

$P^0_{i,i} = 1 > 0$

2) if $i \longleftrightarrow j$, then $j \longleftrightarrow i$

3) if $i \longleftrightarrow k$ and $k \longleftrightarrow j$,

then $i \longleftrightarrow j$ (transitivity)

(Generalization of "=" for $\mathbb{R}$

1) $x = x$

2) if $x = y$, then $y = x$

3) if $x = y$ and $y = z$, then $x = z$)