IEOR 4106 lec 6

communication classes

More on recurrence, transience

\[ S = \bigcup_{k=1}^{\infty} C_k \]

(countably infinite)

disjoint communication classes

positive/null recurrence
Recall \( i, j \in \mathbb{F} \)

If \( p_i(\pi) > 0 \), \( p_j(\pi) > 0 \)

for some \( n \geq 0 \)

1) all states communicate with themselves
   \( i \leftrightarrow i \) \( (p_i^0 = 1) \)

2) symmetry: if \( i \leftrightarrow j \) then \( j \leftrightarrow i \)

3) transitivity: if \( i \leftrightarrow k \) and \( k \leftrightarrow j \)
   then \( i \leftrightarrow j \) \( (p_i^m j \geq p_i^m k \cdot p_{k}^m j > 0) \)
Generalization of “=” for $\mathbb{R}$

1) $x = x$, $x \in \mathbb{R}$

2) if $x = y$, then $y = x$

3) if $x = y$ and $y = z$, then $x = z$

$\leftrightarrow$ "equivalence relation"
Communication Classes

Every state space $S$ of a MC can be uniquely written as

$$S = \bigcup_{k=1}^{\infty} C_k \quad \text{(finite or countably infinite union)}$$

of sets $C_k$ called communication classes for which

1) they are disjoint subsets
2) all states within any class communicate but don't with any states in another class.
Examples

Rat in closed maze

\[ S = \{1, 2, 3, 4\} \]

All states communicate

\[ C = S \]

An example of an "irreducible" MC
open maze

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
3 & 6 & 4 \\
0 & & \\
\end{array}
\]

\[S = \{0, 1, 2, 3, 4\}\]

No state 1, 2, 3, or 4 is reachable from 0.

So \[C_1 = \{0\}\]

\[C_2 = \{1, 2, 3, 4\}\]

\[C_1 \cup C_2 = S\]
Gambler's Ruin MC
\[ S = \{0, 1, 2, \ldots, N\} \]

\[ P_{00} = P_{NN} = 1 \]

\[ C_1 = \{0\} \]
\[ C_2 = \{N\} \]
\[ C_3 = \{1, 2, \ldots, N-1\} \]
\[ C, u < z < c_s = S \]
\[ \bar{\mathcal{S}} = \{0, 1, 2, 3\} \]

\[ P = \begin{bmatrix}
    0 & 1 & 2 & 3 \\
    \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
    \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
    \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} \\
    0 & 0 & 0 & 1
\end{bmatrix} \]

\[ \begin{align*}
    \mathcal{C}_1 &= \{0, 1\} \\
    \mathcal{C}_2 &= \{2\} \\
    \mathcal{C}_3 &= \{3\}
\end{align*} \]

\[ \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 = \bar{\mathcal{S}} \]
Simple Random Walk

$0 < p < 1$

$p_{i,j}^{i+1} = p$

$p_{i,j}^{i-1} = 1 - p = q$

$i \in \mathbb{Z}$

$p_{i,j}^{(1)} = p^{4} > 0$

$p_{i,j}^{(2)} = q^{4} > 0$

$p_{i,j}^{(n)} = \frac{1}{j-i} \quad n = |i-j|$

irreducible MC for all $p$. $C = \mathbb{S} = \mathbb{Z}$
State $i \in S$ is either recurrent ($f_i = 1$) or transient ($f_i < 1$)

$$T_{i,c} = \min \{ n \geq 1 : X_n = i \mid X_0 = c \}$$

- $\infty$ if no return to $i$

$$f_i = P(T_{i,c} < \infty)$$

$N_i$ is the total number of visits to $i$:

$$P(N_i = k) = f_i^{k-1} (1 - f_i), \quad k \geq 1$$

Geometric distribution.
\[ N_i = \sum_{n=0}^{\infty} \mathbb{I}(X_n = i \mid X_0 = i) \]

\[ \mathbb{P}(N_i < \infty) = 1 \quad \text{if} \quad f_i < 1 \]
\[ \mathbb{P}(N_i = \infty) = 1 \quad \text{if} \quad f_i = 1 \]

\[ \mathbb{E}(N_i) = \frac{1}{1-f_i} = \left\{ \begin{array} {c} \infty \quad \text{if} \quad f_i = 1 \\ < \infty \quad \text{if} \quad f_i < 1 \end{array} \right. \]
If \( i \leftrightarrow j \) and \( i \) is recurrent then \( j \) is recurrent

\[ P_{ii} > 0 \]

If \( i \leftrightarrow j \) and \( i \) is transient then \( j \) is transient

\[ \Rightarrow \text{ all states in a communication class } \]

either all are recurrent or all transient
Simple Random Walk $0 < p < 1$

irreducible

$C = \emptyset$

$\Rightarrow$ all states together are recurrent or all are transient

$p = \frac{1}{2}$ recurrent

$p \neq \frac{1}{2}$ transient
Rat in closed wise (irreducible)
\[ C = \{ 1, 2, 3, 4 \} \]

Since \( |S| < \infty \)

must be recurrent

\[ \text{all irreducible finite state MCS} \]

are recurrent.
If $i$ is recurrent

then $(\Pr(T_{ij} < \infty) = 1)$

with

9) $E(T_{jj}) < \infty$  Positive recurrent

b) $E(T_{jj}) = \infty$  Null recurrent

Consider a rv

$\Pr(X = k) = \frac{\lambda}{k^2}, \quad k \geq 1, \quad \Pr(X < \infty) = 1$

$c^{-1} = \sum_{k=1}^{\infty} \frac{1}{k} < \infty, \quad E(X) = \sum_{k=1}^{\infty} \frac{c}{k} = \infty$
If $i \leftrightarrow j$ then
  
  if $i$ is pos. rec. then
  
  So is $j$

$\Rightarrow$ all states in a communication class are
either

1) pos. rec.

2) Null. rec.

3) transient

both recurrent
Examples

\[ \begin{array}{c|c|c|c}
1 & 2 & 3 & 4 \\
1 & 4 & 2 & 1 \\
3 & 4 & 1 & 1 \\
3 & 4 & 1 & 1 \\
\end{array} \]

\[ \text{open} \quad \text{closed} \]

\[ C_1 = \{1, 2, 3, 4\} \quad \text{transient} \]

\[ C_2 = \{0\} \quad \text{recurrent} \]

\[ \overline{\tau}_{00} = 1 \]

\[ C = \{1, 2, 3, 4\} \quad \text{recurrent (Positive)} \]
Simple symmetric RW

Recurrent (Null recurrent)

$E(T_{0,0}) = \infty$

$T_{0,0}$ is an example of a RV $X$ such that $\Pr(X < \infty) = 1$

but $E(X) = \infty$.\[1\]
\[ \pi_j \overset{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}(X_n = j \mid X_0 = i) \]

We want \( \pi_j \) to not depend on \( X_0 = i \in \mathcal{S} \) and \( \pi_j > 0 \), \( j \in \mathcal{S} \), and \( \sum_{j \in \mathcal{S}} \pi_j = 1 \). \( \pi \) is a probability distribution.

In the long-run proportion of time, the chain spends in state \( j \).
If \( \Omega \) exists, we can take expected value of both sides to set

\[
E(\Xi \{ X_n = j \mid X_0 = i \}) = p^{(n)}_{i,j}
\]

For all \( i \in \mathcal{A} \)

\[
\Pi_j = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \pi_{i,j}^{(n)}
\]

for all \( i \in \mathcal{A} \)

\( \Pi = (\Pi_j)_{j \in \mathcal{A}} \) row vector
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} r^n = \left( \frac{r}{1-r} \right) \]
If a Markov chain is irreducible and all states are positive recurrent, then:

\[ \pi_j = \frac{1}{\sum \{ \pi_i \}} \]

for all \( j \).

\( \pi \) is called the limiting or stationary distribution of the MC.

\( \sum \pi_j = 1 \)

Positive recurrent:

\[ 1 \leq E(T_{ij}) < \infty \]

\( \pi = (\pi_i) \), i.e.,
if the chain (irreducible) and
is \text{ null rec } \text{ or }
\text{ transient } \text{ then }
\lim_{t \to \infty} \Pr[X_t = i | X_0 = 0] = 0;
No limiting prob. dist. exists
$\text{Pos. rec.}$

$\mathbb{P}(\text{?}) \equiv \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$

$E(T_{ij}) = 4$

$1 \leq j \leq 4$
for an irreducible MC
it is pos. rec. iff
there exists a prob. solution
\( \left( \Pi_j > 0 \quad i \in A \right) \)
\( \sum \Pi_j = 1 \)
\( \Pi_j = \prod_j P \)

to the set of linear equations

in which case
\( \Pi \) is the limiting diet row vector

\( (\Pi_j = \frac{1}{E(C_i,j)}) \) (unique)
\[ P = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & \frac{1}{2} & \frac{1}{2} & 0 \\
3 & 0 & 0 & \frac{1}{2} \\
4 & 0 & \frac{1}{2} & 0
\end{bmatrix} \]

\[ \Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4) \]

\[ \Pi = \Pi \cdot P \]

\[ \Pi_1 = \frac{1}{2} \Pi_2 + \frac{1}{2} \Pi_3 \]

\[ \Pi_2 = \frac{1}{2} \Pi_1 + \frac{1}{2} \Pi_4 \]

\[ \Pi_3 = \frac{1}{2} \Pi_1 + \frac{1}{2} \Pi_4 \]

\[ \Pi_4 = \frac{1}{2} \Pi_2 + \frac{1}{2} \Pi_3 \]

\[ \Pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \text{ is the Solution} \]