1) More on irreducible
   Positive recurrent
   Markov chains

2) Why \( \pi \Pi \) is called a
   Stationary distribution
Recall that if a MC $(X_n)$ is irreducible, then if all states are positive recurrent (\(P(\tau_{ij} < \infty) = 1\) and \(E(\tau_{jj}) < \infty\)) then a limiting probability dist. \(\pi = (\pi_j)\) exists and \(\pi_j = \frac{1}{E(\tau_{jj})}\) \(j \in S\).
\[ \pi_j \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} X_m = \begin{cases} j & | X_0 = i \end{cases} \]

is a long-run proportion of time the chain spends (visits) in state \( j \).

This constant does not depend on the initial state \( X_0 = i \in S \).

And \( \pi_j > 0 \), \( i \in S \),

\[ \sum_{j \in S} \pi_j = 1 \]

is a probability distribution.
Rates for understanding $TT_j$

$TT_j = \text{long-run prop. of time the chain enters state } j$

$\quad = \text{long-run prop. of time the chain leaves state } j$

$\quad = \text{rate at which the chain enters state } j$

$\quad = \text{rate at which the chain leaves state } j$

from Basic Principles not requiring "Markov"
rate out of $i$

$\pi_j = \pi_i \cdot P_{ij}$

rate into $j$

$\sum_{i \in S} \pi_i \cdot P_{ij}$ (Markov property)

Matrix form

$\pi = \pi \cdot P$

For MCS

($\pi$ is a row vector)
\[ \pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)} \]

\[ \mathbb{E}(I\{X_m = j \mid X_0 = i\}) = p_{ij}^{(m)} \]

\[ \prod_{i=1}^{n} \pi_i = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \prod_{i=1}^{n} p_i^{(m)} \]

Each row is \( \pi_i = (\pi_{ij}) \)
multiply both sides of (2) by $P$

\[
\prod_{n=1}^{N} P = \left( \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} P_{m+1} \right) + \left( \lim_{n \to \infty} \left[ \frac{1}{n} \sum_{m=1}^{n} P_{m} \right] \right) + \left( \frac{P_{n+1} - P_{n}}{n} \right)
\]

\[\Rightarrow \quad \prod_{n=1}^{N} P = \prod_{n=1}^{N}\]
for an irreducible $MC (X_0)$ it is positive rec.

if and only if there exists a prob. soln. $\Pi$

to $\Pi = \Pi P$ ($\Pi_j = \sum \Pi_i P_{ij}$, $i \in A$, $j \in A$

in which case $\Pi$ is the limiting dist.

$\left( \Pi_j = \frac{1}{E(C(x_j))}, j \in A \right)$
Proof is by showing that no limit dist.
or $\Pi = \Pi P$ prob. soln.
can exist when chain is transient or null vec.; hence must be pos. vec.

(See Lecture Notes) for proof
If the state space $S$ is finite ($|S| < \infty$) then an irreducible MC is always pos. rec.

Proof: We already know that a chain can only be either positive or null recurrent (transient not possible); so we
must show that

null rec. is not possible (see lecture notes for proof)

So, for irreducible finite state
MCS we can always try to
find \( \Pi \) via \( \Pi \geq \Pi P \)

it always has a prob. soln.

\( \Pi_j > 0, j \in \mathbb{E} \)

\( \sum \Pi_j = 1 \)
Examples: \( S = \{0, 1\} \), \( p = \{0.4, 0.6\} \)

\[
\Pi = (\Pi_0, \Pi_1) \\
\Pi = \Pi \cdot P
\]

\[
\Pi_0 = 0.4 \Pi_0 + 0.7 \Pi_1 \\
\Pi_1 = 0.6 \Pi_0 + 0.3 \Pi_1
\]

\[
\Pi_0 + \Pi_1 = 1
\]

\[
\Pi_0 = \frac{4}{13}, \quad \Pi_1 = \frac{6}{13}
\]

\[
\Pi_1 = \frac{0.6 \Pi_0}{0.7} \\
\Pi_0 + \Pi_1 = 1 \\
\Pi_0 \left(1 + \frac{0.6}{0.7}\right) = 1
\]
\[ \delta = \{ 0, 1 \} \]

\[ 0 < p < 1 \]

\[ 0 < s < 1 \]

\[ P = \sum_{s} \frac{d}{s+1-p} \]

\[ \Pi = \Pi P \]

\[ \Pi_0 = a \Pi_0 + s \Pi_1 \]

\[ \Pi_1 = (1-a) \Pi_0 - (1-s) \Pi_1 \]

\[ \Pi_0 + \Pi_1 = 1 \]

\[ \Pi_1 = \frac{1-p}{s+1-p} \]

\[ \Pi_0 = \frac{1}{1 + \frac{1-p}{s}} = \frac{s}{s+1-p} \]
\[ S = 90.125 \]

\[ P = \begin{bmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{bmatrix} \]

\[ \pi = \pi P \]

\[ \pi_0 = .5\pi_0 + 0.3\pi_1 + 0.2\pi_2 \]
\[ \pi_1 = .4\pi_0 + .4\pi_1 + .3\pi_2 \]
\[ \pi_2 = .1\pi_0 + .3\pi_1 + .5\pi_2 \]

\[ \pi_0 + \pi_1 + \pi_2 \geq 1 \]

\[ S_0 \implies \pi = \left( \frac{2.1}{62}, \frac{2.3}{62}, \frac{1.6}{62} \right) \]

\[ E(T_{90}) = \frac{1}{\pi_0} \]
\[ = \frac{62}{21} \]
When a pos. rec. MC $Q(x_i)$ has been solved for $\Pi$, we then set for free,

$$E(\Pi_{jj}) = \frac{1}{\Pi_{jj}}, \quad j \in \mathcal{I}$$

Expected return times

$$\Pi_{jj}^{\nu_{in}} = \frac{1}{E(\Pi_{jj})}$$
Quick proof that the simple symmetric RW $(p = \frac{1}{2})$

is **null** versus pos. recurrent

Suppose it was pos. rec., then \( \pi \) exists (Prob. limiting dist.)

and \( \pi_j = \frac{1}{\text{ECT}_{jj}} \geq 0 \), \( j \in S \)

\( \sum \pi_j = 1 \)
impossible because for each \( j \in \mathbb{D} \)

\[
T_{ij} \text{ has the same dist. (hence same mean) \( \bar{E}(T_{ij}) \) a constant}
\]

\[\Rightarrow \pi_j \equiv c > 0 \quad j \in \mathbb{D}\]

\[\Rightarrow 1 = \sum_{j=-\infty}^{\infty} \pi_j = \sum_{j=-\infty}^{\infty} c = \infty \quad \Rightarrow \text{Contradiction, null recurrence}\]
Example of a Poincaré rec. MC with infinite state space

Take the simple random walk restricted to stay in $\mathcal{S} = \{0, 1, 2, \ldots\}$

via $P_{0,0} = 2$, $P_{0,1} = 1$

$P_{ij,i+1} = p$

$P_{ij,i-1} = 2$

(i > 1)
Clearly irreducible

let's try solving \( \Pi = \Pi P \)

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
3 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\Pi_0 = \pi_1 \Pi_0 + \pi_2 \Pi_1 \quad \Rightarrow \quad \pi_0 = \frac{3 \pi_1}{1-\varepsilon} = \rho
\]

\[
\Pi_1 = \rho \Pi_0 + \varepsilon \Pi_2 \quad \Rightarrow \quad \Pi_1 = 3 \pi_1 + \varepsilon \pi_2
\]
\[
\prod_{j} \Pi_{j} = \sum_{j=0}^{\infty} \Pi_{j+1} \quad j \geq 0
\]

\[
\Pi_{1} = \left( \frac{p}{3} \right)^{2} \Pi_{0}
\]

\[
\Pi_{2} = \left( \frac{p}{3} \right)^{2} \Pi_{1} = \left( \frac{p}{3} \right)^{3} \Pi_{0}
\]

\[
\Pi_{j} = \left( \frac{p}{3} \right)^{j} \Pi_{0}, \quad j \geq 0
\]

\[
\sum \Pi_{j} = 1 \iff \sum_{j=0}^{\infty} \left( \frac{p}{3} \right)^{j} = 1
\]
$2 > p$ so that 

\[ \frac{p}{3} \leq 1. \]

\[ \prod_{j=0} = 1 - \frac{p}{3} \]

\[ \prod_{j} = \left( \frac{p}{3} \right)^j \left( 1 - \frac{p}{3} \right) \]

Geometric dist. with mass at 0, $\prod_0 > 0$.
Once we have solved \( \Pi \) for a pos. r.e.c. MC, we might wish to compute

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n = \sum_{i=1}^{J} \pi_i
\]

long-run average = mean of the limiting dist.
\[ \frac{1}{N} \sum_{n=1}^{N} X_n = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} I \{X_n = j\} \]

Let \[ N_j(n) = \sum_{n=1}^{N} I \{X_n = j\} \]

\[ \lim_{N \to \infty} \frac{N_j(n)}{N} = \mathbb{P}_j \text{ by def.} \]

if \((X_n)\) is i.i.d. it holds
More generally suppose for some function $f$ you want

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n)$$

$$= \sum_{j \in \Phi} f(\eta_j) \Pi_j$$

$$\begin{cases} f(x) = x & \text{if } f \geq 0 \text{ or } \sum_{j \in \Phi} |f(\Pi_j)| < \infty \\ f(x) = x^2 & \text{if } f \text{ is bounded or } \sum_{j \in \Phi} |f(\Pi_j)| < \infty \end{cases}$$
Stationarity

If \((X_n)\) is pos. rec. with limiting dist. \(\pi = (\pi_j)\)

and if the chain is started off initially distributed as \(\pi\)

\[\Pr(X_0 = j) = \pi_j, \quad j \in \mathcal{E}\]

Then for each \(n \geq 1\)

\[\Pr(X_n = j) = \pi_j, \quad j \in \mathcal{E}\]
Proof: \[
\text{Suppose } P(X_0 = i) = \pi_i, \quad i \in S
\]

\[
P(X_1 = j) = \sum_{i \in S} P(X_1 = j \mid X_0 = i) \pi_i
\]

\[
= \prod_j \pi_i
\]
Continuing by induction on $n \geq 1$,

$$P(X_{n+1} = j) = \sum_{i \in \mathcal{A}} P(X_{n+1} = j \mid X_n = i) \pi_i.$$

$$= \sum_{i \in \mathcal{A}} \pi_i \cdot P_{ij} = \pi_j.$$

$X_n$ has same dist. $\pi$ for all $n \geq 0$. 


If $X_0 \overset{\text{dist}}{\rightarrow} \pi$, then $(X_n)$ is called a stationary version of the MC.
\[ 8 = \left( \frac{1 + \sqrt{5}}{2} \right)^2 \]

\[ \pi = \left( \frac{1}{2} \right)^2 \sqrt{2} \]

\[ P = \left( \frac{1}{2} \right)^2 \sqrt{2} \]

\[ \{X_n\} = \left\{ \frac{1}{2} \right\} \text{ if } x_0 = 1 \]

\[ \{X_n\} = \left\{ \frac{1}{2} \right\} \text{ if } x_0 = 2 \]
Let \( \{X_n^*\} \) denote \( g \), the stationary version.

\[
X_n^* = \begin{cases} 1 & \text{with probability } \frac{1}{3} \\ 2 & \text{with probability } \frac{1}{3} \\ \cdots & \text{with probability } \frac{1}{3} \\ 1 & \text{with probability } \frac{1}{2} \\ 2 & \text{with probability } \frac{1}{2} \end{cases}
\]

for all \( n \).

\[ P(X_n^* = 1) = \frac{1}{2} = P(X_n^* = 2) \]