IEOR 4106 lec 7

1) More on irreducible Positive recurrent Markov chains

2) Why π is called a <u>stationary</u> distribution

Recall that if a MC (X_n) is irreducible, then if all states are positive Recurrent

$$\left(\mathbb{P}(T_{jj} < \infty) = 1 \text{ and } E(T_{jj}) < \infty \right)$$

then a limiting probability dist.

$$\Pi = (\Pi_j)_{j \in \mathcal{S}} \text{ exists}$$

and $\boxed{\Pi_j = \dfrac{1}{E(T_{jj})}}$, $j \in \mathcal{S}$

① $\pi_j \overset{def}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} I\{X_m = j \mid X_0 = i\}$

wp1

$=$ long-run proportion of time the chain spends (visits) in state j

Constant that doesn't depend on the initial state $X_0 = i \in \mathcal{S}$;

and $\pi_j > 0$, $j \in \mathcal{S}$,

$\sum_{j\in\mathcal{S}} \pi_j = 1$ a probability distribution

Rates for understanding π

π_j = long-run prop. of time the chain enters state j

= long-run prop. of time the chain leaves state j

from Basic Principles; not requiring "Markov"

= rate at which the chain enters state j

= rate at which the chain leaves state j rate out of j $\qquad$ rate into j

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{ij} \quad j \in \mathcal{S}$$

(Markov property)

$$\boxed{\text{matrix form} \quad \pi = \pi P}$$

For MCs

(π is a row vector)

$$\pi_j = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P_{ij}^{(m)}$$

$$\left(E\left(I\{X_m = j \mid X_0 = i\}\right) = P_{ij}^{(m)} \right)$$

$$\boxed{\begin{pmatrix} \pi^- \\ \pi \\ \vdots \\ \pi \end{pmatrix} = \boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m}}$$ (2)

$\uparrow$ each row is $\pi = (\pi_j)$

multiply both sides of (2) by P

$$\begin{pmatrix} \Pi \\ \vdots \\ \Pi \end{pmatrix} P = \left(\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} P^{m+1} \right)$$

$$\lim_{n \to \infty} \left[\left(\frac{1}{n} \sum_{m=1}^{n} P^{m} \right) + \underbrace{\frac{(P^{m+1} - P)}{n}}_{\to 0} \right]$$

$$\underbrace{}_{\text{"}\Pi\text{"}}$$

$\rightarrow$ "$\Pi P = \Pi$"

for an irreducible MC (X_n)

it is positive rec.

if and only if there exists

a prob. soln Π

to $\Pi = \Pi P$ $\left(\Pi_j = \sum_{i \in S} \Pi_i P_{ij},\ j \in S\right)$

in which case

Π is the limiting dist,

$\left(\Pi_j = \frac{1}{E(\tau_{jj})},\ j \in S\right)$

Proof is by showing that NO limiting dist. or $\boxed{\pi = \pi P}$ prob. soln. can exist when chain is transient or null rec;

hence must be pos. rec.

(See Lecture Notes for proof)

If the state space $\mathcal{S}$ is finite ($|\mathcal{S}| < \infty$) then an irreducible MC is always pos. rec.

Proof: we already know chain can only be either positive or Null recurrent (transient not possible); so we

must show that
Null rec. is not
possible (See Lecture
Notes for Proof)

So, for irreducible finite state
M'Cs we can always try to
find π via $\pi = \pi P$

it always has a Prob. Soln.
$\pi_j > 0,\ j \in \mathcal{S}$
$\sum_j \pi_j = 1$

Examples: $\mathcal{S} = \{0, 1\}$ $\quad P = \begin{bmatrix} .4 & .6 \\ .7 & .3 \end{bmatrix}$

$$\pi = (\pi_0, \pi_1)$$

$$\pi = \pi P$$

$$\pi_0 = .4\pi_0 + .7\pi_1$$

$$\boxed{\begin{aligned} \pi_1 &= \frac{.6\pi_0}{.7} \\ \pi_0 + \pi_1 &= 1 \\ \pi_0\left(1 + \frac{6}{7}\right) &= 1 \end{aligned}}$$

$$\boxed{\pi_1 = .6\pi_0 + .3\pi_1}$$

$$\pi_0 + \pi_1 = 1$$

$$\boxed{\pi_0 = \frac{7}{13}, \quad \pi_1 = \frac{6}{13}}$$

$S = \{0, 1\}$

$0 < \alpha < 1$

$0 < \beta < 1$

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\pi = \pi P$$

$$\pi_0 = \alpha \pi_0 + \beta \pi_1 \Rightarrow \pi_1 = \frac{(1-\alpha)\pi_0}{\beta}$$

$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

$$\pi_0 \left(1 + \frac{1-\alpha}{\beta}\right) = 1$$

$$\pi_0 + \pi_1 = 1$$

$$\boxed{\pi_0 = \frac{1}{1 + \frac{1-\alpha}{\beta}} = \frac{\beta}{\beta + 1 - \alpha}}$$

$$\boxed{\pi_1 = \frac{1-\alpha}{\beta + 1 - \alpha}}$$

$$S = \{0, 1, 2\}$$

$$P = \begin{bmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{bmatrix} \qquad \pi = \pi P$$

$$\pi_0 = .5\pi_0 + .3\pi_1 + .2\pi_2$$
$$\pi_1 = .4\pi_0 + .4\pi_1 + .3\pi_2$$
$$\pi_2 = .1\pi_0 + .3\pi_1 + .5\pi_2$$
$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solve

$$\pi = \left(\frac{21}{62}, \frac{23}{62}, \frac{18}{62}\right)$$

$$\boxed{E(T_{0,0}) = \frac{1}{\pi_0} = \frac{62}{21}}$$

When a pos. rec. MC (X_n) has been solved for π we then get for free,

$$E(T_{jj}) = \frac{1}{\pi_j} \quad , j \in \mathcal{S}$$

Expected return times via

$$\left(\pi_j = \frac{1}{E(T_{jj})} \right)$$

Quick proof that the simple symmetric RW $(p = \frac{1}{2})$

is <u>Null</u> versus pos. recurrent

Suppose it was <u>pos. rec.</u>

then π exists (prob. limiting dist.)

and $\pi_j = \dfrac{1}{E(T_{jj})} > 0 \quad j \in S$

$\sum \pi_j = 1$

impossible because

for each $j \in \mathcal{S}$

T_{jj} has the same dist. (hence same mean $E(T_{jj})$)

$$\Rightarrow \pi_j \equiv \overset{\text{a constant}}{c} > 0 \qquad j \in \mathcal{S}$$

$$\Rightarrow 1 = \sum_{j=-\infty}^{\infty} \pi_j = \sum_{j=-\infty}^{\infty} c = \infty \qquad \text{contradiction.}$$

$\Rightarrow$ Null recurrence

Example of a pos. rec. MC with infinite state space

Take the Simple Random walk restricted to stay in $\mathcal{S} = \{0, 1, 2, \dots\}$

$\boxed{0 < p < 1}$

via $P_{0,0} = q$, $P_{0,1} = p$

$P_{i,i+1} = p$

$P_{i,i-1} = q$

$i \geq 1$

Clearly irreducible

let's try solving $\pi = \pi P$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \cdots \\ \hline 0 & q & p & 0 & 0 & 0 \\ 1 & q & 0 & 0 & 0 & 0 \\ 2 & 0 & q & 0 & p & 0 \\ 3 & 0 & 0 & q & 0 & p \\ \vdots & & & & & \end{array}$$

$\pi_0 = q\pi_0 + q\pi_1 \Rightarrow p\pi_0 = q\pi_1$ $(1-q = p)$

$\pi_1 = p\pi_0 + q\pi_2 \Rightarrow \pi_1 = q\pi_1 + q\pi_2 \Longrightarrow$

$$\boxed{p\,\pi_j = q\,\pi_{j+1}} \quad j \geq 0$$

$$\pi_1 = \left(p/q\right)\pi_0$$

$$\pi_2 = \left(p/q\right)\pi_1 = \left(p/q\right)^2\pi_0$$

$$\vdots$$

$$\pi_j = \left(p/q\right)^j \pi_0 \quad j \geq 0$$

$$\sum \pi_j = 1 \iff \boxed{\pi_0 \sum_{j=0}^{\infty} \left(p/q\right)^j = 1} \iff$$

$q > p$ so that $p/q < 1$.

$\Pi_0 = 1 - p/q$ (geometric series! $\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}$ for $|q|<1$)

$\Rightarrow \Pi_j = \left(p/q\right)^j \left(1 - p/q\right) \quad j \geq 0$

geometric dist. with mass at 0, $\Pi_0 > 0$

Once we have solved π for a pos. rec. MC we might wish to compute

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n = \sum_{j\in S} j\pi_j$$

long-run average = mean of the limiting dist.

$$\frac{1}{N}\sum_{n=1}^{N} X_n = \boxed{\sum_{j\in\mathcal{S}} j \, \frac{N_j(N)}{N}} \xrightarrow{N\to\infty} \sum_{j\in\mathcal{S}} j\,\pi_j$$

Let $N_j(N) = \sum_{n=1}^{N} I\{X_n = j\}$

$$\lim_{N\to\infty} \frac{N_j(N)}{N} \underset{wp1}{=} \pi_j \quad \text{by def.}$$

if (X_n) is ≥ 0 it holds

More generally suppose for some function f you want

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n)$$

$$\left(\begin{array}{c} f(x)=x \\ f(x)=x^2 \\ f(x)=I\{x\le a\} \\ \text{etc} \end{array} \right)$$

$$= \sum_{j\in \mathcal{S}} f(j)\pi_j$$

(if $f \geq 0$ or f is bded or $\sum_{j\in \mathcal{S}} |f(j)|\pi_j < \infty$)

Stationarity

if (X_n) is pos. rec.
with limiting dist. $\Pi = (\Pi_j)$

and if the chain is started off initially distributed as Π

$$\mathbb{P}(X_0 = j) = \Pi_j \quad j \in \mathcal{S}$$

then for each $n \geq 1$, $\mathbb{P}(X_n = j) = \Pi_j$
$j \in \mathcal{S}$.

Proof:

$\boxed{\text{Suppose } \mathbb{P}(X_0 = i) = \pi_i}$, $i \in \mathcal{S}$

$\mathbb{P}(X_1 = j)$

$= \sum_{i \in \mathcal{S}} \mathbb{P}(X_1 = j \mid X_0 = i)\, \pi_i$

$= \sum_{i \in \mathcal{S}} \pi_i p_{ij} = \pi_j$ via "$\pi = \pi P$"

Continuing by induction on $n \geq 1$

$$\mathbb{P}(X_{n+1}=j) = \sum_{i \in S} \underbrace{\mathbb{P}(X_{n+1}=j \mid X_n=i)}\pi_i$$

$$= \sum_{i \in S} \pi_i P_{ij} = \pi_j$$

X_n has $\underline{\underline{\text{same}}}$ dist. π for $\underline{\text{all}}$ $n \geq 0$

If $X_0 \overset{\text{dist.}}{=} \pi$, then

$\{X_n\}$ is called a

stationary version of the

MC.

$$\mathcal{S} = \{1, 2\} \qquad \pi = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \checkmark$$

$$P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$$\{X_n\} = \begin{cases} \{1, 2, 1, 2, \ldots\} & \text{if } X_0 = 1 \\ \{2, 1, 2, 1 \ldots\} & \text{if } X_0 = 2 \end{cases}$$

Let $\{X_n^*\}$ denote a stationary version

$$\{X_n^*\} = \begin{cases} \{1, 2, 1, 2, \dots\} & \text{wp} = \frac{1}{2} \\ \{2, 1, 2, 1, \dots\} & \text{wp} = \frac{1}{2} \end{cases}$$

time n: 0 1 2 3

$\mathbb{P}(X_n^* = 1) = \frac{1}{2} = \mathbb{P}(X_n^* = 2)$ for all n