4106 Lec 8

Stopping times
Strong Markov Property

Wald's Equation
Stopping times \( \tau \)

For a stochastic process \( \{X_n \mid n \geq 0\} \)

a random time \( \tau \) is a random variable on \( \Omega \) and we want to consider \( X_\tau \)
if $\tau = n$, then $X_\tau = X_n$

$X_\tau$ denotes the state of the process at time $\tau$.
Definition:

A **stopping time** for \( \{X_n\} \) is a random time such that for each time \( n \geq 0 \), the event \( \{T = n\} \) is determined by (at most) only \( X_0, \ldots, X_n \).
If a random time $\tau$ is independent of the stochastic process $\{X_n\}_{n \in \mathbb{Z}}$ then it is a special case of a stopping time. For each $n$, the event $\{\tau = n\}$ is independent of $\{X_n\}$, hence depends not at all on $\tau$. 
Examples of stopping times for fixed $i \in \mathcal{X}$

$\tau = \min \{ n \geq 0 : X_n = i \}$

$\tau = 0 \iff \{ X_0 = i \} \checkmark$

$\tau = 1 \iff \{ X_0 \neq i, X_1 = i \} \checkmark$

$\tau = n \iff \{ X_0 \neq i, \ldots, X_{n-1} \neq i, X_n = i \} \checkmark$
Let $A \subseteq \mathcal{S}$

$T = \min \{ n > 0 : X_n \notin A \}$

For example, gamblers run MC

$A = \{ \omega \in \Omega : N \}$

$T$ = time at which the gambler stopped gambling.
$\langle T = 0 \rangle = \{ X_0 \in A \} \checkmark$

$\langle T = n \rangle = \{ X_0 \notin A \wedge \ldots \wedge X_{n-1} \notin A, X_n \in A \} \checkmark$

Example of not a stopping time $j$

"last exit time"

$T = \max \left\{ n \geq 0 \mid X_n = i \right\}$ \(\overline{\text{last time state } i \text{ is visited}}\)

$\langle T = 0 \rangle = \{ X_0 = i \wedge X_1 \neq i \wedge X_2 \neq i \wedge \ldots \} \checkmark$
We know for a MC \( \{X_n\} \)

\[
P(X_{n+1} = j | X_n = i, X_0, \ldots, X_{n-1})
\]

\[
= P(X_{n+1} = j | X_n = i) = P_{ij}
\]

If \( \tau \) is a stopping time, then we also get the "Strong Markov Property"
\[ \mathbb{P}(X_{\tau+t} = j \mid X_{\tau} = i, \{X_n : 0 \leq n < \tau\}) \]

\[ = \rho_{ij} \]

\( \{X_{\tau+n} : n \geq 0 \} \) is the same MC as \( (X_n) \) but started with \( X_0 = X_{\tau} \)
\[ P(X_{n+1} = j) \]

\[ = P(X_{n+1} = j | X_n = i, \{X_m = i_m \text{ for } m \leq n\}, X_0 = i_0) \]

\[ = \rho_{ij} \]
Wald's Equation

Let \((X_n : n \geq 1)\) be any \(\text{iid}\) sequence with \(E|X| < \infty\) and suppose \(T\) is stopping time with \(E(T) < \infty\).

Then \[E \left[ \sum_{n=1}^{T} X_n \right] = E(X) E(T)\]
in general that won’t hold for a random time.

Also if \( \tau \) is independent of \( \{X_n\} \) (a special kind of stopping time) then the proof is easy:

\[
E \left[ \sum_{n=1}^{\tau} X_n \mid \tau = k \right] = E \left[ \sum_{n=1}^{k} X_n \right] = k \ E(X)
\]
$$E \left[ \prod_{n=1}^{\infty} X_n \right] = \sum_{k=1}^{\infty} E(Y_1 \mid T=k) p(T=k)$$

$$= \sum_{k=1}^{\infty} k E(x) p(T=k)$$

$$= E(x) \sum_{k=1}^{\infty} k p(T=k) = E(x) E(T)$$
General Proof:

\[
\mathbb{E}\left[ \sum_{n=1}^{\infty} X_n \right] = \mathbb{E}\left[ \sum_{n=1}^{\infty} X_n I\{n \geq \tau_n\} \right]
\]

\[
= \mathbb{E}\left[ \sum_{n=1}^{\infty} X_n I\{n \geq \tau_n\} \right]
\]

independent for each \( n \geq\) depends on at most \( \tau_n \) via "stopping time"
\[ E(X_n 1_{\sum T > n-1}) \]
\[ = E(X) 1_P(CT > n-1) \]

If we are allowed to interchange \( E \) and \( \sum \), then we set
\[ \sum_{n=1}^{\infty} E(X) 1_P(CT > n-1) \]
\[ = E(X) \sum_{n=0}^{\infty} 1_P(CT > n) \]
\[ = E(X 1_E(T)) \]
Interchange is allowed by Fubini's Thm vig using the conditions:

\[
\begin{align*}
\int |f(y) \, d\lambda(y) & \leq \infty \\
\int \left| \int f(y) \, d\lambda(y) \right| \, dx & \leq \infty
\end{align*}
\]
Example: Suppose $X_n$ = outcome of rolling 1 dice $n$th time

$$E(X) = \frac{6+1}{2} = \frac{7}{2} = 3.5$$

$P(X = i) = \frac{1}{6}$, $1 \leq i \leq 6$

$T = \min \{ n \geq 1 : X_n = 6 \}$ a stopping time

$P(T = k) = \left( \frac{5}{6} \right)^{k-1} \frac{1}{6}, \quad k \geq 1$

$E(T) = 6 < \infty$
\[
E\left( \sum_{n=1}^{\tau} X_n \right) = E(T)E(X) = 6 \cdot 3.5 = 21
\]

If \( \tau = 3 \) then we know that \( X_1 \neq 6, X_2 \neq 6, X_3 = 6 \)

The values of \( X_k \) for \( k < n \) are biased.
\[ E\left( \sum_{n=1}^{T} X_n \right) = E(T-1) E(X|X \neq 6) + 6 \]

\[ E(T-1) = E(T)|_{T-1} = 6 - 1 = 5 \]

\[ E(X|X \neq 6) \]

(i.i.d. uniform over \( \{1, 2, 3, 4, 5\} \))

\[ E(X|X \neq 6) = 3 = \frac{5 + 1}{2} \]

\[ 6 + 5 \times 3 = 21 \]
Application to Proving Null recurrence of the Simple symmetric \((p = \frac{1}{2})\) Random Walk

We know all states are recurrent

\[ \mathbb{P}(T_{jj} < \infty) = 1 \quad j \in \mathbb{Z} \]

We also proved \[ E(T_{jj}) = \infty \]

null rec, using other method.

Now we will re-prove.
Sufficient to prove \( E(T_{y1}) = \infty \)

\[
E\left[ T_{y1} \mid \Delta_1 = 1 \right] \geq \frac{1}{2} \left( 1 + E(T_{2y1}) \right)^{1/2} + E\left[ T_{y1} \mid \Delta_1 = -1 \right]^{1/2}
\]

\[
= 1 + \frac{1}{2} E(T_{2y1}) + \frac{1}{2} E(T_{0y1})
\]
it thus suffices to show that
\[ E(T) = \infty \]

where \( T = \min \{ n \geq 1 : R_n = 1 \mid R_0 = 0 \} \)

\[ = T_{01} \text{ a stopping time for } \Delta_1, \Delta_2, \ldots \]

\[ \left( \operatorname{E} |\Delta| = 1 < \infty \right) \]

\[ \sum_{n=1}^{\infty} \lambda_n \]
If $E(T) < \infty$, then we can use Wald's Equation:

$$I = E\left( \sum_{n=1}^{T} \Delta Y_{n} \right) = E(T)E(\Delta Y)$$

Contradiction, hence $E(T) = \infty$
Expected number of visits of a finite state MC to a transient state \( i \) for example the \( E(\text{Total number of visits to room 2 by the rat}) \left\vert X_0 = 1 \right. \) (open maze)
Suppose a Markov chain \((X_n)\)

has finite state space with \(N\) states

\[ A = \begin{pmatrix} 1 & 2 & \cdots & N \end{pmatrix} \]

and there are \(1 \leq b < N\) transient states

set of transient states \( T = \left \{ 1, 2, \ldots, b \right \} \) (\(N-b\) hence recurrent states)

\[ S_{i,j} = \mathbb{E} \left[ \text{Total # visits of } (X_n) \text{ to state } j \mid X_0 = i \right] \]

\(i, j \in T\)
\[ S_{ij} = E \left[ \sum_{n=0}^{\infty} I_{X_n=j \mid X_0=i} \right] \]

\[
P_T \overset{\text{def}}{=} (P_{ij})_{i,j \in T}
\]

\[
S = (S_{ij})_{i,j \in T}
\]

is also a \( b \times b \) matrix

\( b \times b \) matrix (not stochastic)
Prop. Let $I$ denote the $b \times b$ identity matrix.

Then

$$S = I + PS$$

$$\Rightarrow S = (I - P_T)^{-1}$$

We will prove at next lecture.