1) Expected number of visits to transient states

2) Binomial lattice model for risky assets. Introduction to option pricing theory.

(See Lecture Notes 6 on course website)
Computing

\[ S_{i,j} = E \left[ \text{number of visits to state } j \mid X_0 = i \right] \]

\[ S_{i,j} = E \left[ \sum_{n=0}^{\infty} \mathbb{I}(X_n = j \mid X_0 = i) \right] \]

finite = \{y_2, \ldots, N\}

1 \leq b < N \text{ transient states}

\[ T = \sum_{j \neq \text{finite}} \]

(N-b recurrent states left over)
\[ S = (S_{ij})_{i,j \in T} \]

\[ P = (P_{ij})_{i,j \in T} \]

6x6 matrix

Not a stochastic matrix; some rows will not sum to 1

\[ S_{ii} \geq 1 \quad (\text{visits to } i) \]

\[ N_i \text{ = total number of visits to } i \mid X_0 = i \]

\[ i \in T \]

\[ f_i = \text{prob}(T_{ii} < \infty) \leq 1 \]

\[ f_i^{k-1} (1 - f_i) \]
\[ N_i = \sum_{n=0}^{\infty} \mathbb{I}\{X_n = i \mid X_0 = i\} \]

\[ \mathbb{E}(N_i) = E(S_{ii}) \]

\[ \frac{1}{1 - f_i} \]

(But at this point, we do not know how to compute either \( f_i \) or \( E(S_{ii}) \))
Prop 1.1  Let $I$ denote the $b \times b$ identity matrix.

\[
I = \begin{bmatrix}
1 & 0 \\
0 & \ddots & \ddots \\
0 & \ddots & \ddots & 0 \\
\end{bmatrix}
\]

Then

\[
S = I + p_T S
\]

\[
\Rightarrow S - p_T S = I
\]

\[
\Rightarrow (I - p_T) S = I
\]

\[
\Rightarrow S = (I - p_T)^{-1}
\]
Recall: $\text{Det}(AB) = \text{Det}(A) \text{Det}(B)$

$\Rightarrow \text{Det} \left( (I - P_T)(S) \right) = \text{Det}(I) = 1 \neq 0$

\[ \Rightarrow \text{Det}(I - P_T) \text{ Det}(S) \]

\[ \Rightarrow \text{both} \neq 0 \]

\[ \Rightarrow (I - P_T)^{-1} \text{ exists.} \]
Case 1: \( j = i \)

\[
S_{ii} = 1 + \sum_{k \in T} P_{ik} S_{ki}
\]

\[\text{Markov Property}\]

\( j \neq i \)

\[
S_{ij} = \sum_{k \in T} P_{ik} S_{kj}
\]

Matrix form

\[
S = I + P_T S
\]
Example

Rat in open maze

\[ T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \]

\[ b = 4 \]

\[ I - P_T = \begin{bmatrix}
1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & -\frac{1}{2} & 1
\end{bmatrix} \]
\[ S = (I - P)^{-1} = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 3 & 3 \\
3 & 3.5 & 2.5 & 3 \\
4 & 2 & 2 & 2 & 3
\end{bmatrix} \]

\[ S_{14} = 3, \quad S_{11} = 4 \]

\[ S_{11} + S_{12} + S_{13} + S_{14} = 13 \]

\[ = E(T_{10}) \]

\[ = E(\text{escape time}) \quad (x_0 = 1) \]

as we computed before using other methods.
Gamblers' Ruin problem

\( \mathcal{A} = \{ 0, 1, 2, \ldots, N \} \)

\( \mathcal{T} = \{ 1, 2, \ldots, N-1 \} \)

\( S_{ij} = \mathbb{E}(\# \text{ gambles leading to } X_n = i \text{ before game ends} \mid X_0 = i) \)

Sum of row \( i \) = \( \mathbb{E}(\# \text{ gambles until stopping} \mid X_0 = i) \)

Suppose \( N = 10 \), \( i = 3 \)

\( S_{3,0} = \mathbb{E}(\# \text{ times gambler had } \$ 0 \text{ before stopping} \mid X_0 = 3) \)
What about computing

\[ f_i = \text{IP}(T_{ii} < \infty), \ i \in T \ ? \]

More generally:

\[ \text{IP}(X_n \text{ will ever visit } j \mid X_0 = i) \]

\[ S_{ii} = \frac{1}{1 - f_i} \]

\[ f_i = \frac{S_{ii} - 1}{S_{ii}} \]
\[ S_{ij} = f_{ii} S_{jj} + (1 - f_{ii})(0) \]

\[ f_{ii} = \frac{S_{ii}}{S_{jj}} \]

Rat in open maze

\[ f_{23} = \frac{S_{23}}{S_{33}} = \frac{2.5}{3.5} = 0.7 \]

\( f_{34} = 1 \)