IEOR 4106 lec 9

1) Expected number of visits to transient states

2) Binomial lattice model for risky assets.
Introduction to option pricing theory
(See Lecture Notes 6 on course website)

Computing

$$S_{ij} = E\left[\text{Number of visits to state } j \mid X_0 = i\right] \quad i,j \in T$$

$$S_{ij} = E\left[\sum_{n=0}^{\infty} I\{X_n = j \mid X_0 = i\}\right]$$

$$= \sum_{n=0}^{\infty} P_{ij}^{(n)}$$

$\mathcal{S}$ finite $= \{1, 2, \dots, N\}$

$|\mathcal{S}| = N$

$1 \le b < N$ transient states

$T = \{1, \dots, b\}$

($N - b$ recurrent states left over)

$$S = (S_{ij})_{i,j \in T}$$ 6×6 matrix

$$P_T = (P_{ij})_{i,j \in T}$$ 6×6 matrix

Not a <u>stochastic</u> matrix; some rows will not sum to 1

$$S_{ii} \geq 1 \quad (\text{via } X_0 = i)$$

N_i = Total number of visits to $i \mid X_0 = i$, $i \in T$

$$\mathbb{P}(N_i = k) = f_i^{k-1}(1 - f_i)$$

, $f_i = \mathbb{P}(\tau_{ii} < \infty) < 1$

$$N_i = \sum_{n=0}^{\infty} I\{X_n = i \mid X_0 = i\}$$

$$E(N_i) = E(S_{ii})$$

?

$$= \frac{1}{1-f_i}$$

(but at this point, we do not know how to compute either f_i or $E(S_{ii})$)

Prop 1.1 Let I denote the 6×6 identity matrix

$$I = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{bmatrix}$$

then $\boxed{S = I + P_T S}$

$$\Rightarrow \quad S - P_T S = I$$

$$\Rightarrow \quad \underbrace{(I - P_T)}_{M} S = I \quad \Rightarrow \quad S = (I - P_T)^{-1}$$

Recall: $\det(AB) = \det(A)\det(B)$

$$\Rightarrow \det\left((I-P_T)(S)\right) = \det(I) = 1 \neq 0$$

$$\| \quad \det(I-P_T)\det(S)$$

$\Rightarrow$ <u>both</u> $\neq 0$

$\Rightarrow (I-P_T)^{-1}$ exists.

Case 1 : $j = i$

$$S_{ii} = 1 + \sum_{k \in T} P_{ik} S_{ki}$$

Markov Property

$j \neq i$

$$S_{ij} = \sum_{k \in T} P_{ik} S_{kj}$$

Matrix form $\boxed{S = I + P_T S}$ ✓

Example

Rat in open maze

$T = \{1, 2, 3, 4\}$

$b = 4$

$\{0\}$ rec state

$$P_T = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 4 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}$$

$$I - P_T = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ 3 & -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 4 & 0 & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}$$

$$S = (I - P_T)^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 4 & 3 & 3 & 3 \\ 3 & 3.5 & 2.5 & 3 \\ 3 & 2.5 & 3.5 & 3 \\ 2 & 2 & 2 & 3 \end{bmatrix} \end{array}$$

$S_{14} = 3$, $S_{11} = 4$

$$S_{11} + S_{12} + S_{13} + S_{14} = 13$$

$= E(T_{1,0})$

$= E($escape time $| X_0 = 1)$

as we computed before using other methods

Gambler's Ruin problem

$$\mathcal{S} = \{0, 1, 2, \cdots, N\}$$

$$T = \{1, 2, \cdots, N-1\}$$

$$S_{ij} = E\left(\begin{array}{c}\text{\# Gambles leading to} \\ \text{fortune} = j \\ \text{before game ends}\end{array} \middle| X_0 = i\right)$$

$$\text{Sum of row } i = E\left(\begin{array}{c}\text{\# gambles} \\ \text{until stopping}\end{array} \middle| X_0 = i\right)$$

Suppose $N = 10$, $i = 3$

$$S_{3,6} = E\left(\begin{array}{c}\text{\# times gambler} \\ \text{had \$6 before stopping}\end{array} \middle| X_0 = 3\right)$$

What about computing

$$f_i = \mathbb{P}(T_{ii} < \infty), \quad i \in T \quad ?$$

more generally $\boxed{f_{ij}} = \mathbb{P}(X_n \text{ will ever visit } j \mid X_0 = i)$

$j \neq i$

$$\boxed{\underbrace{s_{ii}}_{E(N_i)} = \frac{1}{1-f_i}} \Rightarrow \boxed{f_i = \frac{s_{ii}-1}{s_{ii}}}$$

$j \neq i$

$$S_{ij} = f_{ij} S_{jj} + (1 - f_{ij})(0)$$

$$= f_{ij} S_{jj}$$

$$\boxed{f_{ij} = \frac{S_{ij}}{S_{jj}}}$$

Rat in open maze

$$f_{2,3} = \frac{S_{23}}{S_{33}} = \frac{2.5}{3.5} = 5/7$$

$(f_{2,4} = 1)$