1 Non-stationary Poisson processes and Compound (batch) Poisson processes

Assuming that a Poisson process has a fixed and constant rate $\lambda$ over all time limits its applicability. (This is known as a time-stationary or time-homogenous Poisson process, or just simply a stationary Poisson process.) For example, during rush hours, the arrivals/departures of vehicles into/out of Manhattan is at a higher rate than at (say) 2:00AM. To accommodate this, we can allow the rate $\lambda = \lambda(t)$ to be a deterministic function of time $t$. For example, consider time in hours and suppose $\lambda(t) = 100$ per hour except during the time interval (morning rush hour) $(8,9)$ when $\lambda(t) = 200$, that is $\lambda(t) = 200$, $t \in (8,9)$, $\lambda(t) = 100$, $t \notin (8,9)$.

In such a case, for a given rate function $\lambda(s)$, the expected number of arrivals by time $t$ is thus given by

$$m(t) = E(N(t)) = \int_0^t \lambda(s)ds.$$  \hspace{1cm} (1)

For a compound such process such as buses arriving: If independently each bus holds a random number of passengers (generically denoted by $B$) with some probability mass function $P(B = k)$, $k \geq 0$, and mean $E(B)$. Letting $B_1, B_2, \ldots$ denote the iid sequential bus sizes, the number of passengers to arrive by time $t$, $X(t)$ is given by

$$X(t) = \sum_{n=1}^{N(t)} B_n,$$  \hspace{1cm} (2)

where $N(t)$ is the counting process for the non-stationary Poisson process; $N(t)$ = the number of buses to arrive by time $t$. This is known as a compound or batch non-stationary Poisson arrival process. We have $E(X(t)) = E(N(t))E(B) = m(t)E(B)$.

We have already learned how to simulate a stationary Poisson process up to any desired time $t$, and next we will learn how to do so for a non-stationary Poisson process.

1.1 The non-stationary case: Thinning

In general the function $\lambda(t)$ is called the *intensity* of the Poisson process, and the following holds:

*For each $t > 0$, the counting random variable $N(t)$ is Poisson distributed with mean

$$m(t) = \int_0^t \lambda(s)ds.$$*

$$E(N(t)) = m(t)$$

$$P(N(t) = k) = e^{-m(t)} \frac{m(t)^k}{k!}, \ k \geq 0.$$  \hspace{1cm} (3)

*More generally, the increment $N(t + h) - N(t)$ has a Poisson distribution with mean $m(t + h) - m(t) = \int_t^{t+h} \lambda(s)ds.$*
Furthermore, \( \{N(t)\} \) has independent increments;

If \( 0 \leq a < b < c < d \), then \( N(b) - N(a) \) is independent of \( N(d) - N(c) \).

We shall assume that the intensity function is bounded from above: There exists a \( \lambda^* > 0 \) such that

\[ \lambda(t) \leq \lambda^*, \quad t \geq 0. \]

(In practice, we would want to use the smallest such upper bound.) (Note that if \( \lambda(t) \) was unbounded, then with very mild further assumptions (such as continuity, etc.), it would be bounded over any finite time interval \( (0, T) \) and hence our method could be used over any finite time interval anyhow.)

Then the simulation of the Poisson process is accomplished by a “thinning” method: First simulate a stationary Poisson process at rate \( \lambda^* \). For example, sequentially generate iid exponential rate \( \lambda^* \) interarrival times and use the recursion \( v_{n+1} = v_n + \left(-\frac{1}{\lambda^*}\right) \ln (U_{n+1}) \), to obtain the arrival times which we are denoting by \( v_n \). The rate \( \lambda^* \) is larger than needed for our actual process, so for each arrival time \( v_n \), we independently flip a coin to decide whether to keep it or reject it. The sequence of accepted times we denote by \( \{t_n\} \) and forms our desired non-stationary Poisson process. To make this precise: for each arrival time \( v_n \), we accept it with probability

\[ p_n = \frac{\lambda(v_n)}{\lambda^*}, \]

and reject it with probability \( 1 - p_n \). Thus for each \( v_n \) we generate a uniform \( U_n \) and if \( U_n \leq p_n \) we accept \( v_n \) as a point, otherwise we reject it.

The thinning algorithm for simulating a non-stationary Poisson process with intensity \( \lambda(t) \) that is bounded by \( \lambda^* \)

Here is the algorithm for generating our non-stationary Poisson process up to a desired time \( T \) to get the \( N(T) \) arrival times \( t_1, \ldots t_{N(T)} \).

1. \( t = 0, \quad N = 0 \)
2. Generate a \( U \)
3. \( t = t + \left(-\frac{1}{\lambda^*}\right) \ln (U) \). If \( t > T \), then stop.
4. Generate a \( U \).
5. If \( U \leq \frac{\lambda(t)}{\lambda^*} \), then set \( N = N + 1 \) and set \( t_N = t \).
6. Go back to 2.

Note that when the algorithm stops, the value of \( N \) is \( N(T) \) and we have sequentially simulated all the desired arrival times \( t_1, t_2 \ldots \) up to \( t_{N(T)} \).

Here is a proof that this thinning algorithm works:

**Proof**: [Thinning algorithm] Let \( \{M(t)\} \) denote the counting process of the rate \( \lambda^* \) Poisson process. First note that \( \{N(t)\} \) has independent increments since \( \{M(t)\} \) does and the thinning is done independently. So what is left to prove is that for each \( t > 0 \), \( N(t) \) constructed by this thinning has a Poisson distribution with mean \( m(t) = \int_0^t \lambda(s)ds \). We know that for each \( t > 0 \), \( M(t) \) has a Poisson distribution with mean \( \lambda^* t \). We will partition \( M(t) \) into \( N(t) \) (the accepted ones), and \( R(t) \) (the rejected ones), and conclude that \( N(t) \) has the desired Poisson distribution. To this end recall that conditional on \( M(t) = n \), we can treat the \( n \) unordered arrival times as iid uniform \( (0, t) \) rvs. Thus a typical arrival, denoted by \( V \sim Unif(0, t) \), will be accepted with conditional probability \( \lambda(v)/\lambda^* \), conditional on \( V = v \). Thus the unconditional probability of
acceptance is $q(t) = E[\lambda(V)/\lambda^*] = (1/\lambda^*)(1/t) \int_0^t \lambda(s) ds$, and we conclude from partitioning\(^1\) that $N(t)$ has a Poisson distribution with mean $\lambda^* t q(t) = m(t)$, as was to be shown.

\(^1\)A general result in elementary probability is known as partitioning of a Poisson rv: Suppose $X$ is a rv with a Poisson distribution with mean $\alpha$. Suppose that for all $n \geq 1$, conditional on $X = n$, each of the $n$ is independently labeled as being of type 1 or 2 with probability $p, 1 - p$ respectively, and let $X_i$ denote the number of type $i, i = 1, 2$ (in particular $X = X_1 + X_2$). Then the two $X_i$ are independent rvs, and $X_1$ has a Poisson distribution with mean $\alpha p$ and $X_2$ has a Poisson distribution with mean $\alpha(1 - p)$.

1.2 Simulating a compound Poisson process

Suppose that we wish to simulate a non-stationary compound Poisson process at rate $\lambda(t) \leq \lambda^*$ with iid $B_i$ distributed as (say) $G$ (could be continuous or discrete). Suppose that we already have an algorithm for generating from $G$.

Here is the algorithm for generating our compound Poisson process up to a desired time $T$ to get $X(T)$:

1. $t = 0, N = 0, X = 0.$
2. Generate $U$
3. $t = t + [-(1/\lambda^*) \ln(U)].$ If $t > T$, then stop.
4. Generate $U$
5. If $U \leq \lambda(t)/\lambda^*$, then set $N = N + 1$ and generate $B$ distributed as $G$ and set $X = X + B.$
6. Go back to 2.