HMWK 1

1. 3 black balls and 3 white balls are distributed among two urns (labeled 1,2). At each
“move”, a ball is randomly selected from each urn and the two are swapped (interchanged).
Let \( X_n \) denote the number of Black balls in Urn 1 after the \( n^{th} \) move. Argue that \( X_n \)
forms a Markov chain, and give the transition probability matrix \((P_{i,j})\).

Solution: The state space is \( S = \{0,1,2,3\} \). We assume that each urn contains 3 balls
each.

Given \( X_n = i \in S \), we know that the first urn contains \( i \) black \((B)\) balls and \(3 - i\)
white \((W)\) balls, whereas the second urn contains \(3 - i\) \(B\) balls and \(i\) \(W\) balls. We thus
know the complete distribution of the balls among the urns if we know the state of \( X_n \).
Randomly selecting one ball from each urn and swapping them, indeed yields a MC: If
\( X_n = i \in \{0,1,2,3\} \), then, independent of the past, the ball chosen from the first
urn will be \( B \) \( \text{wp} = i/3 \), and \( W \) \( \text{wp} = (3 - i)/3 \), whereas (independently) the ball chosen
from the second urn will be \( B \) \( \text{wp} = (3 - i)/3 \), and \( W \) \( \text{wp} = i/3 \). This leads to the following
transition probabilities:
\[
P_{01} = 1, \quad P_{10} = \frac{1}{9}, \quad P_{21} = \frac{4}{9}, \quad P_{32} = 1, \quad P_{11} = \frac{4}{9}, \quad P_{22} = \frac{4}{9}, \quad P_{12} = \frac{4}{9}, \quad P_{23} = \frac{1}{9}.
\]

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\
0 & \frac{5}{9} & \frac{5}{9} & \frac{1}{9} \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

For example \( P_{10} = 1/3 \times 1/3 = 1/9 \) because this results from the events: Chose \( B \) from
the first urn and chose \( W \) from the second urn. \( P_{22} = \frac{4}{9} \) because either a \( W \) was swapped
with a \( W \) \( \text{wp} = 2/9 \), or a \( B \) was swapped with a \( B \) \( \text{wp} = 2/9 \), yielding a total (sum) of
4/9.

2. Consider modeling the weather where we now assume that the weather today depends
(at most) on the previous three days weather. Letting \( W_n \) denote weather on the \( n^{th} \)
day \((0 = \text{no rain}, 1 = \text{rain})\), let \( X_n = (W_{n-2},W_{n-1},W_n) \). Argue that \( \{X_n\} \)
forms a MC. There are 8 states, and we can relabel them \( 0-7 \) (as was pointed out in lecture).
\((0,0,0) = 0, \ (1,0,0) = 1, \ (0,1,0) = 2, \ (0,0,1) = 3, \ (1,1,0) = 4, \ (1,0,1) = 5, \ (0,1,1) =
6, \ (1,1,1) = 7. \) Assume that if it has rained for the past 3 days, then it will rain today
with probability 0.7; if it did not rain on any of the past three days, then it will rain
today with probability 0.10. In any other case assume that the weather today will with
probability 0.8 be the same as the weather yesterday. Derive the transition matrix.

Solution: A state is in order (left to right) of “two days ago” “yesterday”, “today”. For
instance, \((1,0,x)\), where \( x = 0,1; \) so it becomes either \((1,0,0)\) or \((1,0,1)\). So there are only two
possibilities for each transition. This means that each row of the transition matrix will
have only two non-zero elements.

We re-label the states \( 0-7 \). The info given, for example, “if it has rained for the past 3
days, then it will rain today with probability 0.7” means that \( P(X_{n+1} = (1,1,1) \mid X_n =
(1,1,1)) = 0.7 \) and \( P(X_{n+1} = (1,1,0) \mid X_n = (1,1,1)) = 0.3 \). In our re-labeling this
becomes \( P(X_{n+1} = 7 \mid X_n = 7) = 0.7 \) and \( P(X_{n+1} = 4 \mid X_n = 7) = 0.3 \), yielding the last
row of the matrix below. Similarly, “if it did not rain on any of the past three days, then
it will rain today with probability 0.10” yields \( P(X_{n+1} = (0,0,1) \mid X_n = (0,0,0)) = 0.1 \)
and \( P(X_{n+1} = (0,0,0) \mid X_n = (0,0,0)) = 0.9 \), or \( P(X_{n+1} = 3 \mid X_n = 0) = 0.1 \) and \( P(X_{n+1} = 0 \mid X_n = 0) = 0.9 \); yielding the initial row of the matrix. The others rows are derived in a similar way, each with a .8 and a .2 element.

\[
P = \begin{pmatrix}
(000) & (100) & (010) & (001) & (110) & (101) & (011) & (111) \\
(000) & .9 & 0 & 0 & .1 & 0 & 0 & 0 \\
(100) & .8 & 0 & 0 & .2 & 0 & 0 & 0 \\
(010) & 0 & .8 & 0 & 0 & .2 & 0 & 0 \\
(001) & 0 & 0 & .2 & 0 & 0 & 0 & .8 \\
(110) & 0 & 0 & 0 & .2 & 0 & 0 & 0 \\
(101) & 0 & 0 & .2 & 0 & 0 & 0 & .8 \\
(011) & 0 & 0 & 0 & 0 & .2 & 0 & .8 \\
(111) & 0 & 0 & 0 & 0 & .3 & 0 & 0 & .7
\end{pmatrix}
\]

### Solution

#### 3. For the MC, \( S_{n+1} = S_n Y_{n+1} \) \( n \geq 0 \) where \( \{Y_n : n \geq 1\} \) forms an iid sequence of rvs, \( P(Y = u) = p \), \( P(Y = d) = 1 - p \), where \( 0 < d < 1 < u \), find the value of \( p \) for which \( E(S_n) = S_0 \) \( n \geq 0 \): the value of \( p \) for which the price, on average, remains constant over time.

**Solution:** By the independence of the \( Y_i \), it follows that \( E(Y_1 Y_2 \cdots Y_n) = E(Y_1) E(Y_2) \cdots E(Y_n) = E(Y)^n \); thus \( E(S_n) = S_0 E(Y)^n \), and we conclude that \( E(S_n) = S_0 \) if and only if \( E(Y) = 1 \). Thus we need to find \( p \) such that

\[ E(Y) = pu + (1 - p)d = 1. \]

Solving yields \( p = (1 - d)/(u - d) \).

#### 4. Consider the rat in the maze problem from class (4 cells). Suppose that the maze is closed so that the rat can never escape: Whenever the rat enters cell 4, it will return to cell 2 or 3 one move later with probability \( 1/2 \). \( (P_{42} = P_{43} = 1/2) \). Thus the rat forever wanders around the maze.

(a) Given that \( X_0 = 1 \), compute the probability that the rat is in cell 3 after his 3rd move.

**Solution:** We want \( P(X_3 = 3 \mid X_0 = 1) = P_{1,3}^{(3)} \), a 3-step transition probability. From Chapman-Kolmogorov, we know that it suffices to compute the matrix \( P^{(3)} = P^3 = (P_{i,j}^{(3)}) \), where \( P \) is the 1-step transition matrix given above, and simply read off the element \( P_{1,3}^{(3)} \). Doing so we obtain

\[
P^3 = P = \begin{pmatrix}
0 & 1/2 & 1/2 & 0 \\
1/2 & 0 & 0 & 1/2 \\
1/2 & 0 & 0 & 1/2 \\
0 & 1/2 & 1/2 & 0
\end{pmatrix}
\]

yielding \( P_{1,3}^{(3)} = 1/2 \). Of course in this example we can also compute \( P_{1,3}^{(3)} \) directly by noting that there are exactly 4 possible paths: \( 1 \to 2 \to 1 \to 3, 1 \to 3 \to 4 \to 3, 1 \to 3 \to 1 \to 3, \) and \( 1 \to 2 \to 4 \to 3 \); each occuring with probability 1/8 for a sum of 1/2.

(b) Given that the rat starts off in cell 1 \( (X_0 = 1) \) find the expected number of moves until it first enters cell 4. Formally: Let \( \tau_{14} = \min\{n \geq 0 : X_n = 4 \mid X_0 = 1\} \). Find \( E(\tau_{14}) \). (Note that \( \tau_{44} = 0 \).)
Solution: Conditioning on the first step \( X_1 = j \) (given \( X_0 = 1, 2, 3 \) respectively) yields three linear equations with three unknowns

\[
\begin{align*}
E(\tau_{14}) &= \frac{1}{2} E(\tau_{24}) + \frac{1}{2} E(\tau_{34}) + 1 \\
E(\tau_{24}) &= \frac{1}{2} E(\tau_{14}) + \frac{1}{2} (1) \\
E(\tau_{34}) &= \frac{1}{2} E(\tau_{14}) + \frac{1}{2} (1)
\end{align*}
\]

Solving yields

\[
\begin{align*}
E(\tau_{14}) &= 4 \\
E(\tau_{24}) &= 3 \\
E(\tau_{34}) &= 3
\end{align*}
\]

5. Let \( \{Y_n : n \geq 0\} \) be an i.i.d. sequence of r.v.s. with \( a_j = P(Y = j) > 0, -\infty < j < \infty \). Define

\[
m_n = \min\{Y_0, \ldots, Y_n\}.
\]

From Lecture Notes 2 we know that \( \{m_n\} \) forms a Markov chain.

(a) The state space is infinite on both sides, \( S = \{\cdots, -2, -1, 0, 1, 2, \cdots\} \). The transition probabilities for \( m_n \) are computed as \( P_{ij} = P(\min(i, Y) = j) \) while considering the 3 cases:

\[
\begin{align*}
P_{i,j} &= P(Y = j) = a_j \text{ for } j < i \\
P_{i,i} &= P(Y \geq i) = \sum_{k \geq i} a_k \\
P_{i,j} &= 0 \text{ for } i < j \text{ (because } m_n \text{ can never increase; it is a non-increasing sequence)}
\end{align*}
\]

(b) With probability 1 (wp1), \( m_n \rightarrow -\infty \) as \( n \rightarrow \infty \) because of the assumption that \( a_j > 0 \) for all \( j < 0 \): for every \( j < 0 \), no matter how small, there will always appear (wp1, for \( n \) large enough) a \( Y_n \) for which \( Y_n = j \).

(c) In this case \( a_j = 0, j < -3 \) so \( m_n \rightarrow -3 \) as \( n \rightarrow \infty \). The state space is now finite: \( S = \{-3, -2, -1, 0, 1, 2, 3\} \). \( P_{i,j} = P(Y = j) = a_j = 1/7 \) for \(-3 < j < i \leq 3\); \( P_{i,i} = P(Y \geq i) = \sum_{k=i}^{3} a_k = (3 - i + 1)/7 \) for \(-3 \leq i \leq 3 \); \( P_{i,j} = 0 \) for \( i < j \) (because \( m_n \) can never increase)

In matrix form,

\[
P = \begin{bmatrix}
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
-3 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1/7 & 6/7 & 0 & 0 & 0 & 0 \\
-1 & 1/7 & 1/7 & 5/7 & 0 & 0 & 0 \\
0 & 1/7 & 1/7 & 1/7 & 4/7 & 0 & 0 \\
1 & 1/7 & 1/7 & 1/7 & 1/7 & 3/7 & 0 \\
2 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 2/7 \\
3 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7
\end{bmatrix}
\]
6. Let $X_n \overset{\text{def}}{=} Y_{n-1} + Y_n, \ n \geq 1, \ X_0 \overset{\text{def}}{=} 0$, where $\{Y_n : n \geq 0\}$ is an i.i.d sequence of rvs with a 0.5 Bernoulli distribution: $P(Y = 0) = P(Y = 1) = 0.5$. Is $\{X_n\}$ a Markov chain? Either prove it is or show why it is not. Repeat for $X_n \overset{\text{def}}{=} Y_{n-1}Y_n, \ n \geq 1$.

**Solution:** Not a MC: For example, if $X_2 = Y_1 + Y_2 = 1$, then we need to also know which of the two, $Y_1$ or $Y_2$ is 1 in order to predict what $X_3 = Y_2 + Y_3$ will be. $P(Y_2 + Y_3 = 0 | X_2 = 1) = 0$ if $Y_2 = 1$, but $P(Y_1 + Y_2 = 0 | X_1 = 1) = 0.5$ if $Y_2 = 0$. The past, $X_1 = Y_0 + Y_1$ contains information that would be relevant, even if we know $X_2$. (If $X_1 = 0$, then we would know that $Y_1 = 0$ and hence $Y_2 = 1$ if $X_2 = 1$.)

The second example also is not a Markov chain for similar reasons: If $X_2 = Y_1Y_2 = 0$, then we need to also know which of the two, $Y_1$ or $Y_2$ is 0 in order to predict what $X_3 = Y_2Y_3$ will be.

**More challenging problem, not for credit**

1. Given $Y_n$, we know all the values of $X_k$ from $n$ onwards, in particular we know $Y_{n+1}$; the future is contained in the present state, hence is independent of the past given we know the present state. If $x = (x_0, x_1, \ldots)$ denotes a sequence and $f$ the function $f(x) = (x_1, x_2, \ldots)$, then in fact $Y_{n+1} = f(Y_n)$. Thus $Y_n$ can be expressed as a recursion (and even without any i.i.d $\{U_n\}$ sequence); hence $\{Y_n\}$ forms a MC.