IEOR 6711, Solutions to HMWK 1, Professor Sigman

1. For X a non-negative rv, and $n \geq 2$ an integer, prove that

$$E(X^n) = \int_0^\infty nx^{n-1}P(X > x)dx.$$

SOLUTION :

$$X^n = \int_0^X nx^{n-1}dx = \int_0^\infty nx^{n-1}I(x),$$

where $I(x) = I\{X > x\}$ denotes the indicator function (in x) of the event $\{X > x\}$. ($I(x) = 1$, if $X > x$, $I(x) = 0$, if $X \leq x$.) Taking expected values and using the non-negativity to justify bringing the expected value inside the integral (Tonelli's Theorem) yields the result:

$$\begin{aligned} E(X^n) &= E\int_0^\infty nx^{n-1}I(x) \\ &= \int_0^\infty nx^{n-1}E(I(x)) \\ &= \int_0^\infty nx^{n-1}P(X > x)dx. \end{aligned}$$

2. Let $\{Y_n : n \geq 0\}$ be an i.i.d. sequence of r.v.s. and let $a_j \stackrel{\text{def}}{=} P(Y = j)$, $-\infty < j < \infty$. Define

$$m_n \stackrel{\text{def}}{=} \min\{Y_0, \ldots, Y_n\},\ n \geq 0.$$

   (a) Show that $\{m_n\}$ forms a Markov chain by expressing it as a recursion.
   SOLUTION: $m_{n+1} = \min\{m_n, Y_{n+1}\},\ n \geq 0$.

   (b) *The transition probablitities for* m_n are computed as $P(\min(i, Y) = j)$ while considering the 3 cases:
   $P_{i,j} = P(Y = j) = a_j$ for $j < i$;
   $P_{i,i} = P(Y \geq i) = \sum_{k \geq i} a_k$;
   $P_{i,j} = 0$ for $i < j$ (because m_n can never increase)

   (c) With probability 1 (wp1), $m_n \to -\infty$ as $n \to \infty$ because of the assumption that $a_j > 0$ for all $j < 0$: for every $j < 0$, no matter how small, there will always appear (wp1, for n large enough) a Y_n for which $Y_n = j$.

   (d) In this case $a_j = 0, j < -3$ so $m_n \to -3$ as $n \to \infty$. $P_{i,j} = P(Y = j) = a_j = 1/7$ for $-3 < j < i \leq 3$;
   $P_{i,i} = P(Y \geq i) = \sum_{k=i}^{3} a_k = (3 - i + 1)/7$ for $-3 \leq i \leq 3$;
   $P_{i,j} = 0$ for $i < j$ (because m_n can never increase)

   In matrix form,

$$P = \begin{array}{c} \\ (-3) \\ (-2) \\ (-1) \\ (0) \\ (1) \\ (2) \\ (3) \end{array} \begin{array}{c} \begin{array}{ccccccc} (-3) & (-2) & (-1) & (0) & (1) & (2) & (3) \end{array} \\ \left\| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/7 & 6/7 & 0 & 0 & 0 & 0 & 0 \\ 1/7 & 1/7 & 5/7 & 0 & 0 & 0 & 0 \\ 1/7 & 1/7 & 1/7 & 4/7 & 0 & 0 & 0 \\ 1/7 & 1/7 & 1/7 & 1/7 & 3/7 & 0 & 0 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 2/7 & 0 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \end{array} \right\| \end{array}$$

3. Let $X_n \stackrel{\text{def}}{=} Y_{n-1} + Y_n,\ n \geq 1,\ X_0 \stackrel{\text{def}}{=} 0$, where $\{Y_n : n \geq 0\}$ is an iid sequence of rvs with a 0.5 Bernoulli distribution: $P(Y = 0) = P(Y = 1) = 0.5$. Is $\{X_n\}$ a Markov chain? Either prove it is or show why it is not. Repeat for the process $X_n \stackrel{\text{def}}{=} Y_{n-1}Y_n,\ n \geq 1$.

   SOLUTION: Not a MC. For suppose that $X_n = Y_{n-1} + Y_n = 1$. Then either $Y_n = 1$ and $Y_{n-1} = 0$, or $Y_n = 0$ and $Y_{n-1} = 1$. To predict the value of $X_{n+1} = Y_{n+1} + Y_n$, we would need to know which of the two cases ocurred; this we could only determine by also knowing past values such as those of X_{n-1}. For example, if $X_{n-1} = 0$, then $Y_{n-1} = 0$, where as if $X_{n-1} = 2$, then $Y_{n-1} = 1$:

   $P(X_{n+1} = 0 | X_n = 1, X_{n-1} = 0) = P(Y_{n+1} + Y_n = 0 | Y_n = 1) = P(Y_{n+1} = -1) = 0$
   $P(X_{n+1} = 0 | X_n = 1, X_{n-1} = 2) = P(Y_{n+1} + Y_n = 0 | Y_n = 0) = P(Y_{n+1} = 0) = 0.5$
   Hence, given X_n, X_{n+1} is not independent of X_{n-1}; the Markov property is not satisfied.

   Similarly, $X_n \stackrel{\text{def}}{=} Y_{n-1}Y_n$ is not a MC either for the same reasons. For example, suppose $X_n = 0$ and we wish to predict X_{n+1}. If $X_n = 0$ because $Y_n = 0$, then $X_{n+1} = Y_nY_{n+1} = 0$ regardless of the value of Y_{n+1}. Whereas if $X_n = 0$ because $Y_n = 1$ and $Y_{n-1} = 0$, then $X_{n+1} = Y_nY_{n+1} = Y_{n+1}$ is Bernoulli. Thus only knowing X_n is not enough to predict X_{n+1}; knowing past values $X_{n-1}, \ldots$ would tell us the precise values of Y_n and Y_{n-1} hence allowing us to predict the future.

4. *Transient problem:* Consider a finite state space ($|\mathcal{S}| < \infty$) MC with transient states $T = \{1, \ldots, b\},\ b \geq 1$, with $P_T = (P_{i,j}), i, j \in T$, the transition matrix for only these transient states. (Recall that a state i is called transient if $f_i < 1$, where f_i denotes the probability the chain will ever return to state i in the future given that $X_0 = i$. Thus such a state is visited only a finite (but random) number of times and then never again. (The other states are called recurrent; $f_i = 1$; they are visited over and over, always returned to again, an infinite number of times.) Let $s_{i,j}$ = the expected number of times (over all time) that the chain visits state $j \in T$ given $X_0 = i \in T$.

$$s_{i,j} = E\{\sum_{n=0}^{\infty} I\{X_n = j | X_0 = i\}\} = \sum_{n=0}^{\infty} P_{ij}^n,$$

   $S = (s_{i,j})$ is a $b \times b$ matrix. (Note that $s_{j,j} \geq 1$.) (Note that it is not possible for the chain to move from a recurrent state to a transient state, why?)

   (a) Letting I denote the $b \times b$ identity matrix, argue that $S = I + P_TS$ yielding the solution
$$S = (I - P_T)^{-1}. \tag{1}$$

SOLUTION: Observe that it is not possible for the chain to move from a recurrent state r to a transient state for otherwise there would be a positive probability that the chain would never return back to r contradicting its recurrence.
Now suppose $X_0 = i \in T$. We first consider the case $j = i$. Since $X_0 = i$ we already have one such (the initial) visit to i. Thus

$$s_{i,i} = 1 + \sum_{k \in T} P_{i,k} s_{k,i},$$

by conditioning on the first state visited, $X_1 = k \in \mathcal{S}$, where we are using the Markov property as soon as we visit the state k to add on the additional visits to i from k. (If $k \notin T$, then there will be no more visits to i, hence the summation only is over $k \in T$.)
For the case $j \neq i$ we do not have any initial visit, hence

$$s_{i,j} = \sum_{k \in T} P_{i,k} s_{k,i}.$$

In matrix form we obtain the result $S = I + P_T S$, and hence the solution $S = (I - P_T)^{-1}$.

(b) In the Gambler's ruin problem, the MC has state space $\{0, 1, \ldots, N\}$ with two recurrent states 0 and N (they are also absorbing states; $P_{00} = P_{NN} = 1$), and the rest being transient $T = \{1, \ldots, N-1\}$. For example, when $N = 5$, $T = \{1, \ldots, 4\}$, and

$$\mathbf{P}_T = \begin{pmatrix} 0 & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1-p & 0 \end{pmatrix}$$

In this $N = 5$ case, assume that $p = 0.5$ and compute S. What is the expected number of times the chain visits 3 given $X_0 = 1$? Repeat for when $p = 0.75$.
SOLUTION: We are to compute $s_{1,3}$. When $p = 0.5$,

$$I - P_T = \begin{pmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{pmatrix}$$

and

$$(I - P_T)^{-1} = \begin{pmatrix} 1.6 & 1.2 & 0.8 & 0.4 \\ 1.2 & 2.4 & 1.6 & 0.8 \\ 0.8 & 1.6 & 2.4 & 1.2 \\ 0.4 & 0.8 & 1.2 & 1.6 \end{pmatrix}$$

$s_{1,3} = 0.8$.

When $p = 0.75$,

$$I - P_T = \begin{pmatrix} 1 & -0.75 & 0 & 0 \\ -0.25 & 1 & -0.75 & 0 \\ 0 & -0.25 & 1 & -0.75 \\ 0 & 0 & -0.25 & 1 \end{pmatrix}$$

and

$$(I - P_T)^{-1} = \begin{pmatrix} 1.32 & 1.29 & 1.19 & 0.89 \\ 0.43 & 1.71 & 1.58 & 1.19 \\ 0.13 & 0.53 & 1.71 & 1.29 \\ 0.03 & 0.132 & 0.43 & 1.32 \end{pmatrix}$$

$s_{1,3} = 1.19$.

(c) For the rat in the open maze (4 cells with the outside as state 0), compute the expected number of visits to cell 3 given the rat starts initially in cell 2.
SOLUTION: $T = \{1, 2, 3, 4\}$ and

$$P_T = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix}.$$

So

$$S = (I - P_T)^{-1} = \begin{pmatrix} 4 & 3 & 3 & 3 \\ 3 & 3.5 & 2.5 & 3 \\ 3 & 2.5 & 3.5 & 3 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

Here $S_{2,3} = 2.5$

(d) Prove that $I - P_T$ is always invertible (e.g., $(I - P_T)^{-1}$ always exists).
SOLUTION: Since $S = I + P_T S$, we have $S(I - P_T) = I$. But since in general, for any two square matrices, A, B it holds that $det(AB) = det(A)det(B)$ (multiplicative property of determinants), we conclude that both S and $I - P_T$ have non-zero determinant, hence are invertable.

5. *Application of the Gambler's Ruin Problem :* A stock starts off initially at price \$8.00 at the end of a day (day 0), and at the end of each consecutive day, the price goes up by one dollar (with probability 0.7) or down by one dollar (with probability 0.3). What is the probability that the stock will reach \$15.00 before going down to a low of \$2.00?

   SOLUTION: Casting this in the context of the Gambler's Ruin Problem in which you win each gamble with probability $p = 0.7$, and lose with probability $q = 0.3$, we equivalently want to compute the probability of reaching state $N = 13$ before reaching state 0 given that $X_0 = i = 6$ (go up by \$7 before going down by \$6):

   $q/p = 3/7$ and so we want P_i when $i = 6$ and $N = 13$:

$$P_6 = \frac{1 - (\frac{3}{7})^6}{1 - (\frac{3}{7})^{13}} = 0.9938$$

6. *(Continuation)* What is the probability that the stock never goes down to \$1.00 but instead increases in value indefinitely (becomes infinitely rich)?

   SOLUTION:

   Now, $i = 7$, and we want

$$P_7(\infty) = \lim_{N \to \infty} P_7(N) = 1 - (\frac{3}{7})^7 = 0.9973$$

7. *(Continuation)* Answer the same two above questions in the case when the two probabilities 0.7 and 0.3 are reversed.

   SOLUTION: Now $q/p = 7/3$ and

$$P_6 = \frac{1 - (\frac{7}{3})^6}{1 - (\frac{7}{3})^{13}} = 0.0026$$

   and

$$P_7(\infty) = \lim_{N \to \infty} P_7(N) = 0$$

8. *State-dependent gambler's ruin problem:* Consider the gambler's ruin problem when $N = 3$, except now we allow the probability p (of winning a dollar) to depend on the present state. Whenever $X_n = 1$, the probability that the gambler wins a dollar is p_1 (and $1 - p_1$ for losing a dollar). Similarly, Whenever $X_n = 2$, the probability that the gambler wins a dollar is p_2 (and $1 - p_2$ for losing a dollar). Find P_1 and P_2, the probabilities that that gambler's fortune reaches 3 before getting ruined, starting with 1 and 2 respectively. Compute for the case when $p_1 = 0.7$ and $p_2 = 0.3$.

   SOLUTION:

   Conditioing on the outcome of the first gamble yields:

$$\begin{aligned} P_1 &= p_1 P_2 + (1 - p_1) P_0 \\ P_2 &= p_2 P_3 + (1 - p_2) P_1, \end{aligned}$$

   which, since $P_0 = 0$ and $P_3 = 1$, becomes

$$\begin{aligned} P_1 &= p_1 P_2 \\ P_2 &= p_2 + (1 - p_2) P_1. \end{aligned}$$

   Solving yields:

$$\begin{aligned} P_1 &= \frac{p_1 p_2}{1 - p_1(1 - p_2)} \\ P_2 &= \frac{p_2}{1 - p_1(1 - p_2).} \end{aligned}$$

   For the given values of $p_1 = 0.7$ and $p_2 = 0.3$ we obtain :

$$\begin{aligned} P_1 &= 0.4118 \\ P_2 &= 0.5882 \end{aligned}$$

   The above state-dependent analysis easily extends to any N: let p_i be the probability when in state i of winning a dollar, $1 \le i \le N - 1$.

9. Let $F(x),\ x \in \mathbb{R}$, denote any cumulative distribution function (cdf) (continuous or not). Define the generalized inverse of F via

$$F^{-1}(y) = \min\{x : F(x) \ge y\},\ y \in [0, 1].$$

(a) Prove that $F^{-1}(F(x)) \le x$.

SOLUTION:

$F^{-1}(F(x)) = \min\{z : F(z) \ge F(x)\}$.

Now, since $z = x$ already satisfies $F(z) \ge F(x)$, the minimum such z must satisfy $z \le x$ yielding $F^{-1}(F(x)) \le x$ as was to be shown.

Also, don't forget that every cdf $F(x)$ is always non-decreasing and right-continuous:

$F(x_1) \le F(x_2)$ if $x_1 \le x_2$.

$\lim_{0<\epsilon\downarrow 0} F(x+\epsilon) = F(x)$ for all x.

(b) Prove that $F(F^{-1}(y)) \ge y$.

SOLUTION:

Just the definition of $F^{-1}(y)$: it is the minimum value of x such that $F(x) \ge y$.

(c) Define $X = F^{-1}(U)$, where U is a rv with a uniform distribution over the interval $[0,1]$. Show that

$$\{U < F(x)\} \subseteq \{X \le x\} \subseteq \{U \le F(x)\}.$$

SOLUTION:

If $U < F(x)$, then since F^{-1} is a non-decreasing function (easy to check since F is so), we apply F^{-1} to both sides to obtain $X = F^{-1}(U) \le F^{-1}(F(x)) \le x$, where for the last inequality we used (a) above. Thus $\{U < F(x)\} \subseteq \{X \le x\}$. Now if $X \le x$, equivalently, $F^{-1}(U) \le x$, then we apply F to both sides and use (b) above to obtain $U \le F(F^{-1}(U)) \le F(x)$; $U \le F(x)$. Thus $\{X \le x\} \subseteq \{U \le F(x)\}$ and the proof is complete.

(d) *Inverse transform method :* Take probabilities in (c) to show that $P(X \le x) = F(x)$; $X = F^{-1}(U)$ is distributed as F.

SOLUTION:

Since U is continuous, we have $P(U = F(x)) = 0$ so that $P(U < F(x)) = P(U \le F(x)) = F(x)$. Thus taking probabilities in (c) yields $F(x) \le P(X \le x) \le F(x)$.

(e) Consider a discrete rv X with probability mass function (pmf) $P(X = k) = p_k,\ k \ge 0$. Show that in this case, the construction $X = F^{-1}(U)$ is explicitly given by: $X = 0$ if $U \le p_0$,

$$X = k, \text{ if } \sum_{i=0}^{k-1} p_k < U \le \sum_{i=0}^{k} p_k,\ k \ge 1.$$

SOLUTION:

$F(x) = p_0,\ 0 \le x < 1;\ = p_0 + p_1,\ 1 \le x < 2, \cdots,\ = \sum_{n=0}^{k-1} P_n,\ k-1 \le x < k$.

Thus $F^{-1}(y) = 0$ if $u \le p_0$,

$$F^{-1}(y) = k, \text{ if } \sum_{i=0}^{k-1} p_k < y \le \sum_{i=0}^{k} p_k,\ k \ge 1.$$