IEOR 6711, HMWK 1, Professor Sigman

1. For X a non-negative rv, and $n \geq 2$ an integer, prove that

$$E(X^n) = \int_0^\infty nx^{n-1}P(X > x)dx.$$

   HINT: $X^n = \int_0^X nx^{n-1}dx =$

2. Let $\{Y_n : n \geq 0\}$ be an i.i.d. sequence of r.v.s. and let $a_j \stackrel{\text{def}}{=} P(Y = j),\ -\infty < j < \infty$. Define

$$m_n \stackrel{\text{def}}{=} \min\{Y_0, \ldots, Y_n\},\ n \geq 0.$$

   Show that $\{m_n\}$ forms a Markov chain by expressing it as a recursion.

   (a) Determine the transition probabilities $P_{i,j}$ in terms of the a_j.

   (b) Assume that $a_j > 0,\ -\infty < j < \infty$. Compute $\lim_{n\to\infty} m_n$.

   (c) Suppose instead that $a_j = P(Y = j) = 1/7,\ j \in \{-3, -2, -1, 0, 1, 2, 3\};\ a_j = 0$ otherwise. (Thus Y has a discrete uniform distribution over the 7 point set.) Determine the transition probabilities $P_{i,j}$ and find $\lim_{n\to\infty} m_n$.

3. Let $X_n \stackrel{\text{def}}{=} Y_{n-1} + Y_n,\ n \geq 1,\ X_0 \stackrel{\text{def}}{=} 0$, where $\{Y_n : n \geq 0\}$ is an iid sequence of rvs with a 0.5 Bernoulli distribution: $P(Y = 0) = P(Y = 1) = 0.5$. Is $\{X_n\}$ a Markov chain? Either prove it is or show why it is not. Repeat for the process $X_n \stackrel{\text{def}}{=} Y_{n-1}Y_n,\ n \geq 1$.

4. *Transient problem:* Consider a finite state space ($|\mathcal{S}| < \infty$) MC with transient states $T = \{1, \ldots, b\},\ b \geq 1$, with $P_T = (P_{i,j}), i, j \in T$, the transition matrix for only these transient states. (Recall that a state i is called transient if $f_i < 1$, where f_i denotes the probability the chain will ever return to state i in the future given that $X_0 = i$. Thus such a state is visited only a finite (but random) number of times and then never again. (The other states are called recurrent; $f_i = 1$; they are visited over and over, always returned to again, an infinite number of times.) Let $s_{i,j}$ = the expected number of times (over all time) that the chain visits state $j \in T$ given $X_0 = i \in T$.

$$s_{i,j} = E\{\sum_{n=0}^{\infty} I\{X_n = j | X_0 = i\}\} = \sum_{n=0}^{\infty} P_{ij}^n,$$

   $S = (s_{i,j})$ is a $b \times b$ matrix. (Note that $s_{j,j} \geq 1$.) (Note that it is not possible for the chain to move from a recurrent state to a transient state, why?)

   (a) Letting I denote the $b \times b$ identity matrix, argue that $S = I + P_T S$ yielding the solution

$$S = (I - P_T)^{-1}. \tag{1}$$

   (Hint: Condition on the first state visited, $X_1 = j \in \mathcal{S}$, given that $X_0 = i \in T$, and note that it is not possible for the chain to move from a recurrent state to a transient state, why?)

(b) In the Gambler's ruin problem, the MC has state space $\{0, 1, \ldots, N\}$ with two recurrent states 0 and N (they are also absorbing states; $P_{00} = P_{NN} = 1$), and the rest being transient $T = \{1, \ldots, N-1\}$. For example, when $N = 5$, $T = \{1, \ldots, 4\}$, and

$$\mathbf{P}_T = \begin{pmatrix} 0 & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1-p & 0 \end{pmatrix}$$

In this $N = 5$ case, assume that $p = 0.5$ and compute S. What is the expected number of times the chain visits 3 given $X_0 = 1$? Repeat for when $p = 0.75$.

(c) For the rat in the open maze (4 cells with the outside as state 0), compute the expected number of visits to cell 3 given the rat starts initially in cell 2.

(d) Prove that $I - P_T$ is always invertable (e.g., $(I - P_T)^{-1}$ always exists).

5. *Application of the Gambler's Ruin Problem :* A stock starts off initially at price \$8.00 at the end of a day (day 0), and at the end of each consecutive day, the price goes up by one dollar (with probability 0.7) or down by one dollar (with probability 0.3). What is the probability that the stock will reach \$15.00 before going down to a low of \$2.00?

6. *(Continuation)* What is the probability that the stock never goes down to \$1.00 but instead increases in value indefinitely (becomes infinitely rich)?

7. *(Continuation)* Answer the same two above questions in the case when the two probabilities 0.7 and 0.3 are reversed.

8. *State-dependent gambler's ruin problem:* Consider the gambler's ruin problem when $N = 3$, except now we allow the probability p (of winning a dollar) to depend on the present state. Whenever $X_n = 1$, the probability that the gambler wins a dollar is p_1 (and $1 - p_1$ for losing a dollar). Similarly, Whenever $X_n = 2$, the probability that the gambler wins a dollar is p_2 (and $1 - p_2$ for losing a dollar). Find P_1 and P_2, the probabilities that that gambler's fortune reaches 3 before getting ruined, starting with 1 and 2 respectively. Compute for the case when $p_1 = 0.7$ and $p_2 = 0.3$.

9. Let $F(x),\ x \in \mathbb{R}$, denote any cumulative distribution function (cdf) (continuous or not). Define the generalized inverse of F via

$$F^{-1}(y) = \min\{x : F(x) \geq y\},\ \ y \in [0, 1].$$

(a) Prove that $F^{-1}(F(x)) \leq x$.

(b) Prove that $F(F^{-1}(y)) \geq y$.

(c) Define $X = F^{-1}(U)$, where U is a rv with a uniform distribution over the interval $[0, 1]$. Show that

$$\{U < F(x)\} \subseteq \{X \leq x\} \subseteq \{U \leq F(x)\}.$$

(d) *Inverse transform method :* Take probabilities in (c) to show that $P(X \leq x) = F(x)$; $X = F^{-1}(U)$ is distributed as F.

(e) Consider a discrete rv X with probability mass function (pmf) $P(X = k) = p_k,\ k \geq 0$. Show that in this case, the construction $X = F^{-1}(U)$ is explicitely given by: $X = 0$ if $U \leq p_0$,

$$X = k,\ \text{if}\ \sum_{i=0}^{k-1} p_k < U \leq \sum_{i=0}^{k} p_k,\ \ k \geq 1.$$