IEOR 6711, HMWK 2, Professor Sigman

1. For the gambler's ruin problem with $p = 0.60$ and $N = 3$: If the gambler starts off with \$1, then what is the probability that the game ends in no more than 3 gambles? (The game ends when either the gambler reaches N or goes broke, whichever happens first. So we want the probability that the simple random walk MC chain associated with the gambler's ruin problem ($P_{00} = P_{N,N} = 1$), having started with $X_0 = 1$, is either in state 0 or state N at time $n = 3$.)

2. (*Easy proof of Gambler's ruin problem in the symmetric case*) Consider the simple *symmetric* ($p = 0.5$) random walk, $R_n = \Delta_1 + \cdots + \Delta_n$, with $R_0 = i$ where $0 < i < N$. Let $\tau = \min\{n \geq 1 : R_n \in \{0, N\}\}$, the time at which the gambler stops gambling in the gambler's ruin problem.

   (a) Explain why $E(\tau) < \infty$ using elementary MC ideas.

   (b) Derive the identity
   $$E\Big[i + \sum_{n=1}^{\tau} \Delta_n\Big] = NP_i(N).$$

   (c) Use the above identity (justifying along the way) to show that $P_i(N) = i/N$.

3. Consider a man stuck in a cave that has 3 doors: he is equally likely to choose any such door (1/3 probability) with the objective being to reach freedom (escape). If he chooses door 1, he wanders for 5 days and returns to the cave. If he chooses door 2, he wanders for 3 days and returns to the cave. If he chooses door 3, he travels for 2 days and reaches freedom. Whenever choosing the wrong door (1,2) hence returning to the cave, he is (independent of the past) again equally likely to choose any such door.

   (a) Let X denote the total number of days until he reaches freedom. Let D denote the first door he chooses. By conditioning on $D = i,\ i = 1, 2, 3$, derive the following equation:
   $E(X) = 1/3((5 + E(X)) + (3 + E(X)) + 2)$.

   (b) Solve for $E(X)$.

   (c) Here you are to use Wald's equation to find $E(X)$: Define an iid sequence of rvs $Y_1, Y_2, \cdots$ and a stopping time τ, such that
   $$X = \sum_{n=1}^{\tau} Y_n.$$
   Now use this to compute $E(X)$.

4. Suppose that Y_1 and Y_2 are *identically distributed* rvs on the non-negative integers, and their joint distribution is $P(Y_1 = k, Y_2 = l) = a_{k,l},\ k, l \geq 0$. Show how to construct a stationary sequence $\{X_n : n \geq 0\}$ of rvs, such that the joint distribution (X_n, X_{n+1}) is the same as that of (Y_1, Y_2) for all $n \geq 0$. (By stationary, we mean in particular that X_n has the same distribution for all $n \geq 0$.)

5. Recall the geometric series
$$(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n,$$

for $|x| < 1$.

Let $M = (m_{ij})$ be a $b \times b$ matrix and suppose that $M^n \to 0$ as $n \to \infty$ (where here 0 denotes the 0 matrix, and the convergence means that $m_{ij}^{(n)} \to 0$ for each fixed pair i, j). Then

$$(1 - M)^{-1} = \sum_{n=0}^{\infty} M^n. \tag{1}$$

This is so because

$$(I + M + M^2 + M^3 + \cdots + M^n)(I - M) = I - M^{n+1},$$

and if $M^n \to 0$ then as $n \to \infty$, the right hand side converges to I yielding the result.

Let $M = P_T$, the transient states matrix from Homework 1, Problem 4. Argue that $P_T^n \to 0$. Hint: use Chapman-Kolmogorov to see that $P_T^{(n)} = P_T^n$ just as $P^{(n)} = P^n$, and observe that $m_{ij}^{(n)} = p_{i,j}^{(n)} = P(X_n = j \mid X_0 = i)$.

(1) is useful since we can thus approximate $(I - P_T)^{-1}$ by choosing a large k and using

$$(I - P_T)^{-1} \approx \sum_{n=0}^{k} M^n.$$

6. Consider an irreducible Markov chain $\{X_n\}$ with finite state space $\mathcal{S} = \{1, 2, \ldots N\}$, and transition matrix P. Consider the set of N equations, $\pi = \pi P$, and the additional equation $\pi_1 + \cdots + \pi_N = 1$. Thus a total of $N + 1$ equations. Show that if you remove any one of the first N equations, then in fact, the remaining N equations still uniquely determine the stationary distribution π (e.g., they have a unique probability solution that IS the stationary distribution π).

7. A Markov chain transition matrix P is called *doubly stochastic* if its *columns* each sum to 1. Thus each row and each column are non-negative elements summing to 1. Show that if a Markov chain has a finite state space $\mathcal{S} = \{1, 2, \ldots, b\}$ and is both irreducible and doubly stochastic, then the stationary distribution π is the discrete uniform distribution over the set $\mathcal{S}$; $\pi_j = 1/b,\ j \in \{1, 2, \ldots, b\}$.

8. Let $Y_n = \sum_{k=1}^{n} D_k,\ n \geq 1$, where the $\{D_k\}$ are iid with a discrete uniform distribution over the set $\{1, 2, 3, 4, 5, 6\}$ (e.g., the outcome of rolling a dice).

   Find

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{n} P(Y_n = \text{a multiple of } 13).$$

   (HINT: Use the above problem concerning doubly stochastic.)