

IEOR 6711, HMWK 3, Professor Sigman

1. Consider a positive recurrent Markov chain with limiting stationary distribution π . We know that π_i is, by definition, the long-run proportion of time that the chain moves into state i ; $\pi_i = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n I\{X_k = i\}$. It is also a rate: The long-run rate (number of times per unit time) that the chain moves into state i .
 - (a) Argue that π_i is also the long-run proportion of time (rate) that the chain moves out of state i .
 - (b) Argue that $\pi_i P_{i,j}$ is the long-run rate that the chain moves from state i into state j .
 - (c) Argue that

$$\sum_i \pi_i P_{i,j}$$

is the long-run rate that the chain moves into state j .

- (d) Use the above to conclude that, for each j ,

$$\pi_j = \sum_i \pi_i P_{i,j};$$

that is, $\pi = \pi P$.

2. George has $r \geq 3$ umbrellas distributed between home and office as follows: When departing home at the beginning of a day, if it is raining, then he takes an umbrella (if there is one) with him to the office. Similarly, when departing the office at the end of a day, if it is raining, then he takes an umbrella (if there is one) with him to home. Assume that independent of the past there is a fixed probability $0 \leq p \leq 1$ that it rains any time he departs a location (home or office).
 - (a) Argue that $X_n =$ the number of umbrellas at the current location JUST BEFORE he departs for the n^{th} time forms a MC, with state space $\mathcal{S} = \{0, 1, 2, 3, \dots, r\}$ and find the transition matrix. (For example, if $r = 3$, then $P_{2,2} = p$, $P_{2,1} = 1 - p$; $P_{0,3} = 1$ and so on.)
 - (b) Explain why the stationary distribution π must exist for this chain and be unique (e.g., name a Theorem, etc.).
 - (c) Show that π is given by $\pi_0 = (1 - p)/(r + 1 - p)$, $\pi_i = (r + 1 - p)^{-1}$, $1 \leq i \leq r$.
 - (d) George gets wet if and only if he departs a location when it is raining and all the umbrellas are at the other location. What is the long-run proportion of times that George gets wet?
 - (e) What value of p maximizes the long-run proportion of times that George gets wet? Compute for the case when $r = 3$.

3. *Martingale MC*: Consider the MC with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 3/5 & 0 & 1/5 & 1/5 \\ 1/10 & 1/10 & 1/10 & 1/10 & 6/10 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- (a) Show that $E(X_{n+1}|X_n = i) = i$, $i \in \mathcal{S}$. (e.g., each row i of P has mean i .) Thus, being so for each i , we conclude that $E(X_{n+1}|X_n) = X_n$: Given the present state X_n , the expected value of the future is X_n itself.
- (b) Now use the additional fact that $\{X_n\}$ is a MC to deduce that

$$E(X_{n+1}|X_n, X_{n-1}, \dots, X_0) = X_n, \quad n \geq 0 :$$

Given the present state, X_n , the expected value of the future is X_n itself, *independent of the past*.

This is an example of a *martingale*, a stochastic process for which $E(X_{n+1}|X_n, X_{n-1}, \dots, X_0) = X_n$, $n \geq 0$. It models a fair game in gambling: Let X_n denote the total fortune after your n^{th} gamble. Each consecutive gamble on average yields no profit and no loss given the current total earnings, independent of the past. The net earning on the n^{th} gamble is the increment $\Delta_n = X_n - X_{n-1}$, and we can re-write $X_n = X_0 + \Delta_1 + \dots + \Delta_n$, $n \geq 0$. Then the Martingale property equivalently states that $E(\Delta_{n+1}|X_n, X_{n-1}, \dots, X_0) = 0$, $n \geq 0$. Using the general fact that $E[E(X|Y)] = E(X)$ for any Y , we see that $E(\Delta_n) = 0$, $n \geq 1$ and $E(X_n) = X_0$, $n \geq 0$. Thus a martingale X_n is like a symmetric random walk (mean 0 increments), except the increments Δ_i need not be iid.

4. *Infinite dimensional Markov chain.* Let $\{X_n : n \geq 0\}$ be any stochastic process, and let $\mathbf{Y}_0 = \{X_n : n \geq 0\}$, $\mathbf{Y}_1 = \{X_{1+n} : n \geq 0\} = \{X_1, X_2, \dots\}$, \dots , $\mathbf{Y}_k = \{X_{k+n} : n \geq 0\} = \{X_k, X_{k+1}, \dots\}$. Each \mathbf{Y}_k is thus the entire infinite sequence from time k onwards. Argue that in fact $\{\mathbf{Y}_k : k \geq 0\}$ satisfies the Markov property: The future \mathbf{Y}_{k+1} is independent of the past $\mathbf{Y}_0, \mathbf{Y}_1, \dots, \mathbf{Y}_{k-1}$ given the present state \mathbf{Y}_k . Thus, in principle, any stochastic process can be modeled as a Markov chain by including enough information (the infinite future). (But, the complexity is so high that this is of little practical value.)
5. Consider a chess board (64 squares). A knight is initially in the upper left corner and is the only piece on the board (recall that a knight can only move in “L” shaped moves). Assume that every unit of time the knight moves by randomly choosing from its available legal moves in an “equally likely” fashion (independent of any past moves). What is the expected amount of time (moves) until the knight returns back to the upper left corner?
6. Consider the Gambler’s ruin Markov chain $\{X_n\}$ on $\mathcal{S} = \{0, 1, \dots, N\}$. $P_{0,0} = 1 = P_{N,N}$, while otherwise $P_{i,i+1} = p$, $P_{i,i-1} = 1-p$, $1 < i < N$. Suppose that $X_0 = i$, for $1 < i < N$. Let $X(i) = \lim_{n \rightarrow \infty} X_n$. Explain why the rv $X(i)$ must exist (with probability 1), and give its probability mass function: $p_k = P(X(i) = k)$, $k \in \{0, 1, \dots, N\}$.